

MODULI SPACES FOR QUILTED SURFACES AND POISSON STRUCTURES

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ABSTRACT. Let G be a Lie group endowed with a bi-invariant pseudo-Riemannian metric. Then the moduli space of flat connections on a principal G -bundle, $P \rightarrow \Sigma$, over a compact oriented surface, Σ , carries a Poisson structure. If we trivialize P over a finite number of points on $\partial\Sigma$ then the moduli space carries a quasi-Poisson structure instead. Our first result is to describe this quasi-Poisson structure in terms of an intersection form on the fundamental groupoid of the surface, generalizing results of Massuyeau and Turaev [19, 27].

Our second result is to extend this framework to *quilted surfaces*, i.e. surfaces where the structure group varies from region to region and a reduction (or relation) of structure occurs along the borders of the regions, extending results of the second author [25, 23, 24].

We describe the Poisson structure on the moduli space for a quilted surface in terms of an operation on *spin networks*, i.e. graphs immersed in the surface which are endowed with some additional data on their edges and vertices. This extends the results of various authors [13, 12, 22, 4].

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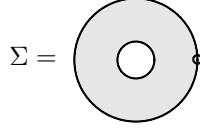
1. INTRODUCTION

If G is a Lie group whose Lie algebra \mathfrak{g} carries an invariant metric, and Σ is a closed oriented surface, the corresponding moduli space of flat connections

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G$$

on principal G -bundles $P \rightarrow \Sigma$ carries a symplectic form [5]; more generally, if Σ has a boundary, then the moduli space carries a Poisson structure.

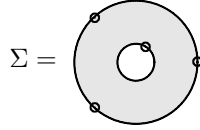
If Σ is connected, and one marks a point on one of the boundary components



and trivializes the principal bundle over that point, the moduli space

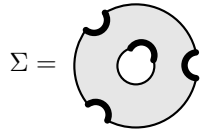
$$\mathrm{Hom}(\pi_1(\Sigma), G)$$

becomes *quasi-Poisson* [3, 2, 1]. In a recent paper [19], Massuyeau and Turaev described this quasi-Poisson structure in terms of an intersection form on the loop algebra $\mathbb{Z}\pi_1(\Sigma)$, extending a result of Goldman [12, 13]. The first result of our paper is to generalize their result to the case where Σ has multiple marked points (possibly on the same boundary component):



These surfaces allow for more economical description of the moduli spaces — in particular, we show how to obtain them from a collection of discs with two marked points each via multiple fusion.

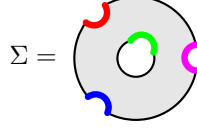
Blowing up at each of the marked points, we obtain a surface which we call a *domain*:



We refer to the preimage of any marked point as a *domain wall* (these are the thickened segments of the boundary in the image above). Our second result is the following: Suppose one chooses a reduction of structure separately for each domain wall \mathbf{w} , i.e.

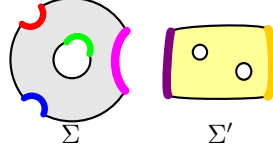
- a subgroup $L_{\mathbf{w}} \subseteq G$, and
- a subbundle $Q_{\mathbf{w}} \rightarrow \mathbf{w}$ of $P|_{\mathbf{w}} \rightarrow \mathbf{w}$ on which $L_{\mathbf{w}}$ acts transitively.

If the Lie algebras $\mathfrak{l}_{\mathbf{w}} \subseteq \mathfrak{g}$ corresponding to $L_{\mathbf{w}} \subseteq G$ each satisfy $\mathfrak{l}_{\mathbf{w}}^{\perp} = \mathfrak{l}_{\mathbf{w}}$, then the moduli space of flat connections on P which are compatible with the reduced bundles $Q_{\mathbf{w}} \rightarrow \mathbf{w}$ is Poisson (this result can be generalized to the case of $\mathfrak{l}_{\mathbf{w}}^{\perp} \subset \mathfrak{l}_{\mathbf{w}}$). We may think of this as ‘coloring’ each domain wall with a reduced structure group $L_{\mathbf{w}} \subseteq G$, as pictured below:



In this way we obtain, in particular, the Poisson structures inverting the symplectic forms carried by the moduli spaces of colored surfaces, introduced in [25] (see also [23, 24]).

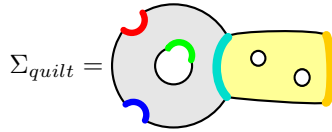
Suppose now that G' is a second Lie group whose Lie algebra \mathfrak{g}' carries an invariant metric, and $P' \rightarrow \Sigma'$ is a principal G' -bundle over a domain Σ' . Once again, we choose a reduction of structure for the bundle $P|_{\mathbf{w}'} \rightarrow \mathbf{w}'$ over each domain wall \mathbf{w}' on Σ' , i.e. a subgroup $L_{\mathbf{w}'} \subseteq G'$. If we simultaneously consider flat connections on P and P' which are compatible with the reduced structure on each domain wall, then (as before) the moduli space is Poisson. We picture this as follows:



However, one might instead wish to choose a common reduction of structure for two domain walls, \mathbf{w} and \mathbf{w}' (on Σ and Σ' , resp.). More precisely, to *sew* the domain walls \mathbf{w} and \mathbf{w}' together is to choose an identification $\phi : \mathbf{w} \rightarrow \mathbf{w}'$, together with

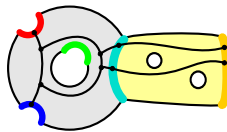
- a subgroup $L_\phi \subseteq G \times G'$, and
- a subbundle $Q_\phi \rightarrow \mathbf{w}$ of $P|_{\mathbf{w}} \times_\phi P'|_{\mathbf{w}'}$ on which L_ϕ acts transitively,

such that the Lie algebra $\mathfrak{l}_\phi \subseteq \mathfrak{g} \oplus \mathfrak{g}'$ corresponding to L_ϕ satisfies $\mathfrak{l}_\phi^\perp = \mathfrak{l}$ (where \mathfrak{g}' denotes the Lie algebra \mathfrak{g}' with the metric negated). Quilted surfaces are surfaces formed by sewing domains together along domain walls, and by choosing a reduction of structure on any of the remaining (unsewn) domain walls, as was previously described.



Such surfaces have played a role in recent developments in both Chern-Simons theory [15, 16, 14] and Floer theory [29, 28]. Our second main result is to show that the moduli space of flat connections, $\mathcal{M}_{\Sigma_{quilt}}$, on a quilted surface is Poisson.

We provide a description of this Poisson structure in terms of *spin networks* [21, 6], as in [22, 4, 13, 12]. More precisely, we identify functions $f \in C^\infty(\mathcal{M}_{\Sigma_{quilt}})$ on the moduli space of flat connections over a quilted surface with spin networks in the quilted surface. Such a spin network $[\Gamma, *]$ consists of an immersed graph $\Gamma \rightarrow \Sigma_{quilt}$,



together with some decoration¹ of the edges and vertices of the graph, which (in the introduction) we will denote abstractly by $*$. The Poisson bracket of two spin networks $[\Gamma, *]$ and $[\Gamma', *']$ is computed as a sum over their intersection points $p \in \Gamma \times_{\Sigma_{quilt}} \Gamma'$,

$$\{[\Gamma, *], [\Gamma', *']\} = \sum_{p \in \Gamma \times_{\Sigma_{quilt}} \Gamma'} \pm [\Gamma \cup_p \Gamma', *''],$$

where $\Gamma \cup_p \Gamma'$ denotes the union of the two graphs with a common vertex added at the intersection point p . This formula generalizes the one found in [22].

The basic technical tool we use is a new type of reduction of quasi-Poisson G -manifolds by subgroups of G .

In this paper we study the moduli spaces from the (quasi-)Poisson point of view. The approach via (quasi-)symplectic 2-forms and a unifying picture using Courant algebroids will appear in a future paper.

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2. QUASI-POISSON MANIFOLDS

In this section we recall the basic definitions from the theory of quasi-Poisson manifolds, as introduced by Alekseev, Kosmann-Schwarzbach, and Meinrenken [2, 1].

Let G be a Lie group with Lie algebra \mathfrak{g} and with a chosen Ad-invariant symmetric quadratic tensor, $s \in S^2 \mathfrak{g}$. Let $\phi \in \bigwedge^3 \mathfrak{g}$ be the Ad-invariant element defined by

$$(1) \quad \phi(\alpha, \beta, \gamma) = \frac{1}{4} \alpha([s^\# \beta, s^\# \gamma]) \quad (\alpha, \beta, \gamma \in \mathfrak{g}^*),$$

where $s^\# : \mathfrak{g}^* \rightarrow \mathfrak{g}$ is given by $\beta(s^\# \alpha) = s(\alpha, \beta)$.

Suppose $\rho : G \times M \rightarrow M$ is an action of G on a manifold M . Abusing notation slightly, we denote the corresponding Lie algebra action $\rho : \mathfrak{g} \rightarrow \Gamma(TM)$, by the same symbol. We extend ρ to a Gerstenhaber algebra morphism $\rho : \bigwedge \mathfrak{g} \rightarrow \Gamma(\bigwedge TM)$.

Definition 1. A quasi-Poisson G -manifold is a triple (M, ρ, π) , where M is a manifold, ρ an action of G on M , and $\pi \in \Gamma(\bigwedge^2 TM)$ a G -invariant bivector field, satisfying

$$\frac{1}{2} [\pi, \pi] = \rho(\phi).$$

This definition depends on the choice of s . If G_1, G_2 are Lie groups with chosen elements $s_i \in S^2 \mathfrak{g}_i$ ($i = 1, 2$), we set $s = s_1 + s_2 \in S^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$, so that we can speak about quasi-Poisson $G_1 \times G_2$ -manifolds. In particular, if (M_i, ρ_i, π_i) is a quasi-Poisson G_i -manifold ($i = 1, 2$) then

$$(M_1, \rho_1, \pi_1) \times (M_2, \rho_2, \pi_2) = (M_1 \times M_2, \rho_1 \times \rho_2, \pi_1 + \pi_2)$$

is a quasi-Poisson $G_1 \times G_2$ -manifold.

Example 1. G is a quasi-Poisson $G \times G$ -manifold, with the action $\rho(g_1, g_2) \cdot g = g_1 g g_2^{-1}$ and with $\pi = 0$.

¹Note, when our structure groups are compact, the graph is decorated with a representation on each edge, and each vertex is decorated with an intertwinor of the representations on the surrounding edges, as in [22].

Remark 1. Since s appears twice in Eq. (1), it follows that any quasi-Poisson (G, s) -manifold is also a quasi-Poisson $(G, -s)$ -manifold. Likewise, any quasi-Poisson $(G_1 \times G_2, s_1 \oplus s_2)$ -manifold is also a quasi-Poisson $(G_1 \times G_2, s_1 \oplus -s_2)$ -manifold.

Let $\psi \in \bigwedge^2(\mathfrak{g} \oplus \mathfrak{g})$ be given by

$$\psi = \frac{1}{2} \sum_{i,j} s^{ij} (e_i, 0) \wedge (0, e_j)$$

where $s = \sum_{i,j} s^{ij} e_i \otimes e_j$ in some basis e_i of \mathfrak{g} .

Definition 2. If (M, ρ, π) is a quasi-Poisson $G \times G \times H$ -manifold then its fusion is the quasi-Poisson $G \times H$ -manifold (M, ρ^*, π^*) , where $\rho^*(g, h) = \rho(g, g, h)$ and

$$\pi^* = \pi - \rho(\psi).$$

Fusion is associative (but not commutative): if M is a quasi-Poisson $G \times G \times G \times H$ -manifold then the two $G \times H$ -quasi-Poisson structures obtained by the double fusion coincide. If M is a quasi-Poisson $G^n \times H$ -manifold then its (multiple) fusion to a quasi-Poisson $G \times H$ -manifold is given by

$$(2) \quad \pi^* = \pi - \sum_{i < j} \rho(\psi_{i,j}),$$

where $\psi_{i,j} \in \bigwedge^2(n \times \mathfrak{g})$ is the image of ψ under the inclusion $\mathfrak{g} \oplus \mathfrak{g} \rightarrow n \times \mathfrak{g}$ sending the two \mathfrak{g} 's to i 'th and j 'th place respectively.

3. REDUCTION AND MOMENT MAPS

A Lie subgroup $C \subseteq G$ will be called *reducing* if its Lie algebra $\mathfrak{c} \subseteq \mathfrak{g}$ satisfies

$$\phi(\alpha, \beta, \gamma) = 0 \quad \forall \alpha, \beta, \gamma \in \text{ann}(\mathfrak{c})$$

where $\text{ann}(\mathfrak{c}) \subseteq \mathfrak{g}^*$ is the annihilator of \mathfrak{c} . Equivalently,

$$[s^\# \alpha, s^\# \beta] \in \mathfrak{c} \quad \forall \alpha, \beta \in \text{ann}(\mathfrak{c}).$$

In particular, if $C \subseteq G$ is coisotropic, i.e. if $s^\#(\text{ann}(\mathfrak{c})) \subseteq \mathfrak{c}$, then C is reducing.

Theorem 1.A. Suppose that (M, ρ, π) is a quasi-Poisson G -manifold and that $C \subseteq G$ is a reducing subgroup. Then

$$(3) \quad \{f, g\} := \pi(df, dg), \quad f, g \in C^\infty(M)^C$$

is a Poisson bracket on the space of C -invariant functions. In particular, if the C -orbits of M form a regular foliation², then the bivector field π descends to define a Poisson structure on M/C .

Proof. The proof is essentially the same as that of [1, Theorem 4.2.2], but we include it for completeness.

First, we observe that $\{f, g\} \in C^\infty(M)^C$, since f, g and π are each C -invariant.

To see that the bracket (3) satisfies the Jacobi identity, notice that

$$\{f_1, \{f_2, f_3\}\} + \text{c.p.} = \frac{1}{2} [\pi, \pi](df_1, df_2, df_3) = \rho(\phi)(df_1, df_2, df_3),$$

for any $f_i \in C^\infty(M)^C$ ($i = 1, 2, 3$). Now $\rho^* df_i \in \text{ann}(\mathfrak{c})$ and C is reducing, hence $\rho(\phi)(df_1, df_2, df_3) = 0$. \square

For $\xi \in \mathfrak{g}$ let ξ^L and ξ^R denote the corresponding left and right invariant vector field on G .

²i.e. the orbit space, M/C , is a manifold, and the projection $M \rightarrow M/C$ is a surjective submersion

Definition 3. Let (M, ρ, π) be a quasi-Poisson G -manifold and let $\tau : G \rightarrow G$ be an s -preserving automorphism. A map $\mu : M \rightarrow G$ is a $(\tau$ -twisted) moment map if it is equivariant for the action $g \cdot \tilde{g} = \tau(g) \tilde{g} g^{-1}$ of G on G , and if the image of π under

$$\mu_* \otimes id : TM \otimes TM \rightarrow TG \otimes TM$$

is

$$-\frac{1}{2} s^{ij} (e_i^L + \tau(e_i)^R) \otimes \rho(e_j).$$

We shall use moment maps to get Poisson submanifolds of M/C , in analogy with Marsden-Weinstein reduction (under certain non-degeneracy conditions these submanifolds will be the symplectic leaves of M/C). First we need an analogue of coadjoint orbits.

Lemma 1. If $C \subseteq G$ is a reducing subgroup then

$$\hat{\mathfrak{c}} := \{(\xi + s^\sharp \alpha, \xi - s^\sharp \alpha); \xi \in \mathfrak{c}, \alpha \in \text{ann}(\mathfrak{c})\}$$

is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$.

Proof. Let $\xi, \eta \in \mathfrak{c}$, $\alpha, \beta \in \text{ann}(\mathfrak{c})$. Since

$$[\xi \pm s^\sharp \alpha, \eta \pm s^\sharp \beta] = ([\xi, \eta] + [s^\sharp \alpha, s^\sharp \beta]) \pm s^\sharp (\text{ad}_\xi^* \beta - \text{ad}_\eta^* \alpha)$$

and $[s^\sharp \alpha, s^\sharp \beta] \in \mathfrak{c}$, the space $\hat{\mathfrak{c}}$ is closed under the Lie bracket. \square

Let $\hat{C} \subseteq G \times G$ be a Lie group with the Lie algebra $\hat{\mathfrak{c}}$. The group $G \times G$ acts on G by

$$(4) \quad (g_1, g_2) \cdot g = \tau(g_1) g g_2^{-1}.$$

The orbits of $\hat{C} \subseteq G \times G$ on G will serve as analogues of coadjoint orbits.

Theorem 1.B. Let (M, ρ, π) be a quasi-Poisson G -manifold with a moment map $\mu : M \rightarrow G$ and $C \subseteq G$ a reducing subgroup. Suppose that the C -orbits of M form a regular foliation. Let $\mathcal{O} \subseteq G$ be a \hat{C} -orbit. If the graph of μ intersects \mathcal{O} cleanly³, then

$$\mu^{-1}(\mathcal{O})/C \subseteq M/C$$

is a Poisson submanifold. More generally, if $K \subseteq G$ is a \hat{C} -stable submanifold and the graph of μ intersects K cleanly, then $\mu^{-1}(K)/C \subseteq M/C$ is a Poisson submanifold.

Proof. First of all notice that $\mu^{-1}(K) \subseteq M$ is stable under the action of $C \subseteq G$.

For any $f \in C^\infty(M)$ the moment map condition gives

$$(5) \quad \mu_*(\pi(\cdot, df)) = -\frac{1}{2} ((s^\sharp \rho^* df)^L + \tau(s^\sharp \rho^* df)^R).$$

If f is C -invariant then $\alpha := \rho^* df$ belongs to $\text{ann}(\mathfrak{c})$. The vector (5) is thus the action of $\frac{1}{2}(s^\sharp \alpha, -s^\sharp \alpha) \in \hat{\mathfrak{c}}$ on G . In particular, $\pi(\cdot, df)$ is tangent to $\mu^{-1}(K)$. This implies that $\mu^{-1}(K)/C \subseteq M/C$ is a Poisson submanifold. \square

The space $\mu^{-1}(\mathcal{O})/C$ can be conveniently described in the following way. Let \hat{M} be the \hat{C} -manifold obtained from M by induction from C to \hat{C} (using the diagonal embedding $C \subset \hat{C}$). Concretely,

$$(6) \quad \hat{M} = (\hat{C} \times M)/C$$

³That is, $\text{gr}(\mu) \cap \mathcal{O} \subseteq G \times M$ is a submanifold, and $T(\text{gr}(\mu) \cap \mathcal{O}) = T \text{gr}(\mu) \cap T \mathcal{O}$, where $\text{gr}(\mu) = \{(\mu(x), x); x \in M\}$. It happens, in particular, if μ is transverse to \mathcal{O} .

where the C -action on $\hat{C} \times M$ is $c \cdot (\hat{c}, m) = (\hat{c}c^{-1}, c \cdot m)$. The moment map $\mu : M \rightarrow G$ induces a \hat{C} -equivariant map

$$\hat{\mu} : \hat{M} \rightarrow G, \quad \hat{\mu}(\hat{c}, m) = \hat{c} \cdot \mu(m).$$

Let now $\mathcal{O} = \hat{C} \cdot g_0$ for some $g_0 \in G$. Since $\mu^{-1}(\mathcal{O})/C \cong \hat{\mu}^{-1}(\mathcal{O})/\hat{C}$, we have

$$(7) \quad \mu^{-1}(\mathcal{O})/C \cong \hat{\mu}^{-1}(g_0)/\text{Stab}(g_0)$$

where

$$\text{Stab}(g_0) = \{\hat{c} \in \hat{C}; \hat{c} \cdot g_0 = g_0\} \subseteq \hat{C}.$$

Partial reduction. One can generalize both Theorem 1.A and Theorem 1.B in order to reduce quasi-Poisson $G \times H$ -manifolds to quasi-Poisson H -manifolds:

Theorem 1.C. *Suppose that (M, ρ, π) is a quasi-Poisson $G \times H$ -manifold, $C \subseteq G$ is a reducing subgroup, and the C -orbits of M form a regular foliation.*

- (1) *The bivector field π descends to define a quasi-Poisson H -structure on M/C .*
- (2) *Let τ_G and τ_H be automorphisms of G and H , and let*

$$(\mu_G, \mu_H) : M \rightarrow G \times H$$

be a (τ_G, τ_H) -twisted moment map. If $K \subseteq G$ is a \hat{C} -stable submanifold and the graph of μ_G intersects K cleanly, then $\mu_G^{-1}(K)/C \subseteq M/C$ is a quasi-Poisson H -submanifold, and μ_H descends to define a τ_H -twisted moment map,

$$\mu_H : \mu_G^{-1}(K)/C \rightarrow H.$$

Proof. The proof of the first statement is only superficially different from that of Theorem 1.A and so we omit it.

Likewise, we omit the proof that $\mu_G^{-1}(K)/C \subseteq M/C$ is a quasi-Poisson H -submanifold, as it differs only superficially from that of Theorem 1.B.

Finally, since $\mu_H : \mu_G^{-1}(K) \rightarrow H$ is $G \times H$ -equivariant, it descends to a map on $\mu_G^{-1}(K)/C$. The image of π under $(\mu_H)_* \otimes \text{id} : TM \otimes TM \rightarrow TH \otimes TM$ is

$$-\frac{1}{2}s_H^{ij}(e_i^L + \tau(e_i)^R) \otimes \rho(e_j),$$

where $s_H \in S^2(\mathfrak{h})^H$ denotes the chosen invariant symmetric tensor, and hence this also holds for the reduced bivector field on both M/C and $\mu_G^{-1}(K)/C$, proving that μ_H descends to define a moment map. \square

4. QUASI-POISSON STRUCTURES ON MODULI SPACES

Let Σ be a compact oriented surface with boundary, and let $V \subset \partial\Sigma$ be a finite collection of marked points such that every component of Σ intersects V . Let $\Pi_1(\Sigma, V)$ denote the fundamental groupoid of Σ with the base set V . The composition in $\Pi_1(\Sigma, V)$ is from right to left: ab means path b followed by path a . For $a \in \Pi_1(\Sigma, V)$ let $\text{out}(a)$ denote the source and $\text{in}(a)$ the target of a ; ab is defined if $\text{in}(b) = \text{out}(a)$.

Let

$$M_{\Sigma, V}(G) = \text{Hom}(\Pi_1(\Sigma, V), G).$$

$M_{\Sigma, V}(G)$ can be seen as the moduli space of flat connections on principal G -bundles over Σ which are trivialized over V .

For any arrow $a \in \Pi_1(\Sigma, V)$ let

$$\text{hol}_a : M_{\Sigma, V}(G) \rightarrow G$$

denote evaluation at a (in terms of flat connections it is the holonomy along a). There is a natural action $\rho = \rho_{\Sigma, V}$ of the group G^V on $M_{\Sigma, V}(G)$ which is defined by

$$(8) \quad \text{hol}_a(\rho(g)x) = g_{\text{in}(a)} \text{hol}_a(x) g_{\text{out}(a)}^{-1}.$$

Infinitesimally,

$$(\text{hol}_a)_*(\rho(\xi)) = -\xi_{\text{in}(a)}^R + \xi_{\text{out}(a)}^L$$

for any $\xi \in \mathfrak{g}^V$, where $\xi^{L/R}$ denotes the left/right invariant vector field on G corresponding to $\xi \in \mathfrak{g}$.

By a *skeleton* of (Σ, V) we mean a graph $\Gamma \subset \Sigma$ with the vertex set V , such that there is a deformation retraction of Σ to Γ .⁴ If we choose an orientation of every edge of Γ then $M_{\Sigma, V}(G)$ gets identified (via $(\text{hol}_a, a \in E_\Gamma)$) with G^{E_Γ} , where E_Γ is the set of edges of Γ . In particular, if Σ is a disc and V has two elements then we get $M_{\Sigma, V}(G) = G$.

The boundary of Σ is split by V to *arcs* (the components of $\partial\Sigma$ that don't contain a marked point⁵ are not considered to be arcs). If we choose an ordered pair (P, Q) of marked points ($P \neq Q \in V$) then the corresponding *fused surface* Σ^* is obtained by gluing a short piece of the arc starting at P with a short piece of the arc ending at Q (so that P and Q get identified). The subset $V^* \subset \partial\Sigma^*$ is obtained from V by identifying P and Q . Notice that the map

$$M_{\Sigma^*, V^*}(G) \rightarrow M_{\Sigma, V}(G),$$

coming from the map $(\Sigma, V) \rightarrow (\Sigma, V^*)$, is a diffeomorphism: if Σ retracts to a skeletal graph Γ then Σ^* retracts to its image Γ^* , and the two graphs have the same number of edges. We can thus identify the manifolds $M_{\Sigma^*, V^*}(G)$ and $M_{\Sigma, V}(G)$.

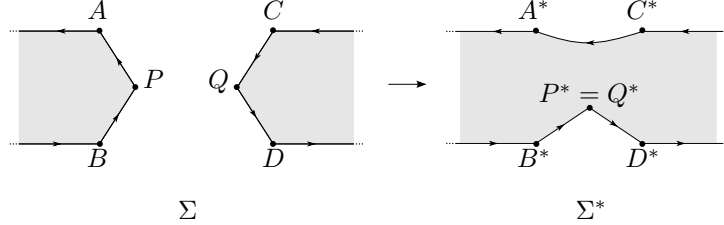


FIGURE 1. Fusion

Every (Σ, V) can be obtained by fusion from a collection of discs, each with two marked points: If $\Gamma \subset \Sigma$ is a skeleton then the subset of Σ that retracts onto an edge $e \in E_\Gamma$ is a disc D_e , and Σ is obtained from D_e 's by repeated fusion.

Theorem 2. *There is a natural bivector field $\pi_{\Sigma, V}$ on $M_{\Sigma, V}(G)$ such that*

$$(M_{\Sigma, V}(G), \rho_{\Sigma, V}, \pi_{\Sigma, V})$$

is a quasi-Poisson manifold, uniquely determined by the properties

- (1) *if Σ is a disc and V has two elements then $\pi_{\Sigma, V} = 0$*
- (2) *if $(\Sigma, V) = (\Sigma_1, V_1) \sqcup (\Sigma_2, V_2)$ then $\pi_{\Sigma, V} = \pi_{\Sigma_1, V_1} + \pi_{\Sigma_2, V_2}$*
- (3) *if (Σ^*, V^*) is obtained from (Σ, V) by fusion, then $(M_{\Sigma^*, V^*}(G), \rho_{\Sigma^*, V^*}, \pi_{\Sigma^*, V^*})$ is obtained from $(M_{\Sigma, V}(G), \rho_{\Sigma, V}, \pi_{\Sigma, V})$ by the corresponding fusion.*

If there is no danger of confusion, we shall denote $\pi_{\Sigma, V}$ simply by π .

⁴It is a simple exercise to show that there exists a skeleton for every marked surface. However, it should be emphasized that this skeleton is not unique!

⁵that is, an element of V

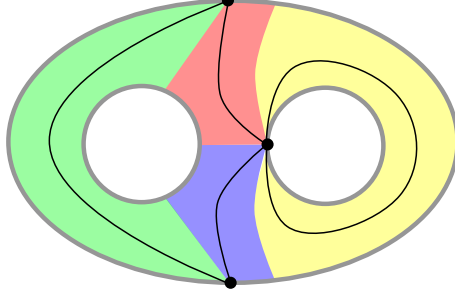


FIGURE 2. A surface with a skeleton, fused from four discs. The discs have been assigned colors in the picture, but this figure should not be confused with a quilted surface.

Remark 2. Alejandro Cabrera has independantly studied quasi-Hamiltonian G^V -structures for the marked surfaces described above.

Once we choose a skeleton Γ of (Σ, V) , Theorem 2 gives us a formula for the quasi-Poisson structure on $M_{\Sigma, V}$, as (Σ, V) is a fusion of a collection of discs with two marked points. Let us denote the resulting bivector field on $M_{\Sigma, V}(G)$ by π_Γ . Theorem 2 follows from the following Lemma:

Lemma 2. *The bivector field π_Γ on $M_{\Sigma, V}(G)$ is independent of the choice of Γ .*

Remark 3. The lemma follows from the special case where (Σ, V) is a disc with 3 marked points (see Example 3 below). However, we shall give a different proof in the next section.

Proof of Theorem 2. By the lemma we have a well-defined quasi-Poisson structure on $M_{\Sigma, V}(G)$. Properties (1)–(3) of the theorem are satisfied by the construction of π_Γ . \square

Let us now describe the calculation of $\pi = \pi_\Gamma$ in more detail. Notice that for any vertex v of Γ , the (half)edges adjacent to v are linearly ordered: a cyclic order is given by the orientation of Σ . Since v is on the boundary, the cyclic order is actually a linear order. Γ is a *ciliated graph* in the terminology of Fock and Rosly [11].

We choose an orientation of every edge of Γ to get an identification $M_{\Sigma, V}(G) = G^{E_\Gamma}$. First we see it as a $G^{E_\Gamma} \times G^{E_\Gamma}$ -quasi-Poisson space with zero bivector (i.e. as $M_{\Sigma', V'}(G)$, where (Σ', V') is a disjoint union of discs with two marked points each). Then, fusing at each vertex using the linear order, we obtain a G^V -quasi-Poisson space.

Example 2. As the simplest example, suppose (Σ, V) is an annulus with a single marked point (on one of the boundary circles). Then (Σ, V) may be obtained by fusion from a disc (Σ', V') with two marked points, as in Fig. 3. Now $M_{\Sigma', V'} = G$ with the the quasi-Poisson $G \times G$ -structure described in Example 1: the bivector field is trivial and $G \times G$ acts by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. Thus $M_{\Sigma, V} = G$, the G -action is by conjugation, and

$$\pi = \frac{1}{2} s^{ij} e_i^R \wedge e_j^L.$$

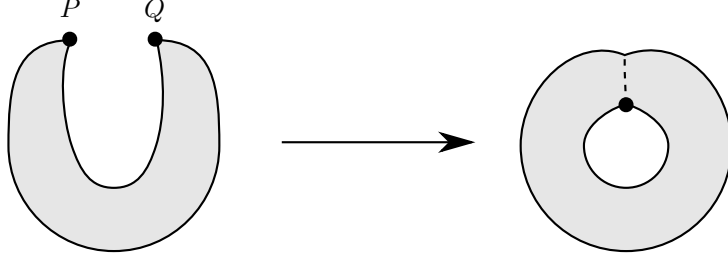
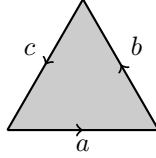


FIGURE 3. The annulus with one marked point is obtained by fusion from the disc with two marked points.

Example 3. Let Σ be a triangle and V is the set of its vertices.



We can identify $M_{\Sigma,V}$ with G^2 via $(\text{hol}_{a^{-1}}, \text{hol}_b)$, i.e. Γ is the graph with the oriented edges a^{-1} , b . In this case

$$(9) \quad \pi = -\frac{1}{2} s^{ij} e_i^L(1) \wedge e_j^L(2)$$

(where $e_i^L(k)$ denotes the left-invariant vector field which is tangent to the k^{th} factor of G^2 ($k = 1, 2$)). Equivalently,

$$\pi(\text{hol}_{a^{-1}}^* \theta^L, \text{hol}_b^* \theta^L) = -s \in \mathfrak{g} \otimes \mathfrak{g},$$

where $\theta^L \in \Omega^1(G, \mathfrak{g})$ is the left-invariant Maurer-Cartan form. An easy calculation shows

$$\pi(\text{hol}_{b^{-1}}^* \theta^L, \text{hol}_c^* \theta^L) = -s \in \mathfrak{g} \otimes \mathfrak{g},$$

confirming that π is independent of the choice of Γ .

For a general surface (Σ, V) with a choice of a skeleton Γ , we get an identification $M_{\Sigma,V} = G^{E_\Gamma}$. Applying (2) we obtain

$$(10) \quad \pi_{\Sigma,V} = -\frac{1}{2} \sum_{v \in V} \sum_{a < b} s^{ij} e_i(a, v) \wedge e_j(b, v)$$

where a, b run over the (half)edges adjacent to v ,

$$e_i(a, v) = \begin{cases} e_i^R(a) & a \text{ goes into } v \\ -e_i^L(a) & a \text{ goes out of } v \end{cases}$$

and for $a \in E_\Gamma$, $e_i^{R,L}(a)$ denotes the right/left-invariant vector field on G^{E_Γ} equal to

$$(0, \dots, 0, \overbrace{e_i}^a, 0, \dots, 0) \in \mathfrak{g}^{E_\Gamma}$$

at the identity element. Essentially the same formula was discovered by Fock and Rosly [11], for Poisson structures on $M_{\Sigma,V}$ obtained by a choice of a classical r -matrix. Meanwhile, Skovborg studied the corresponding formula in the absence of an r -matrix for invariant functions [26].

5. THE HOMOTOPY INTERSECTION FORM AND QUASI-POISSON STRUCTURES

Massuyeau and Turaev [19] made a beautiful observation that, in the case of one marked point and $G = GL_n$, the quasi-Poisson structure on $M_{\Sigma,V}(G)$ can be expressed in terms of the homotopy intersection form on $\pi_1(\Sigma)$, introduced by Turaev in [27]. Here we extend their result to the case of arbitrary (G, s) and arbitrary V .

Let us first extend (a skew-symmetrized version of) Turaev's homotopy intersection form to fundamental groupoids. If $a, b \in \Pi_1(\Sigma, V)$, let us represent them by transverse smooth paths α, β . For any point A in their intersection, let

$$\lambda(A) = \begin{cases} 1 & \text{if } A \in \partial\Sigma \\ 2 & \text{otherwise} \end{cases}$$

$$\text{sign}(A) := \text{sign}(\alpha, \beta; A) = \begin{cases} 1 & \text{if } (\alpha'|_A, \beta'|_A) \text{ is positively oriented} \\ -1 & \text{otherwise.} \end{cases}$$

as in Fig. 4.

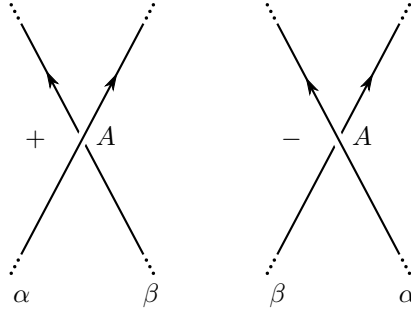


FIGURE 4. $\text{sign}(A) = \pm 1$ is determined by comparing the orientation of α and β with that of Σ .

Let α_A denote the portion of α parametrized from the beginning up to the point A . Finally, let

$$(11) \quad (a, b) := \sum_A \lambda(A) \text{sign}(A) [\alpha_A^{-1} \beta_A] \in \mathbb{Z}\Pi_1(\Sigma, V).$$

As in [27] one can check that (a, b) is well defined, i.e. independent of the choice of α and β .

Let us list the properties of (a, b) . For $x \in \mathbb{Z}\Pi_1(\Sigma, V)$, $x = \sum n_i a_i$, let $\bar{x} = \sum n_i a_i^{-1}$.

Proposition 1. *The map*

$$(\cdot, \cdot) : \Pi_1(\Sigma, V) \times \Pi_1(\Sigma, V) \rightarrow \mathbb{Z}\Pi_1(\Sigma, V)$$

satisfies

- (1) (a, b) is a linear combination of paths from the source of b to the source of a
- (2) $(b, a) = -\overline{(a, b)}$
- (3) $(a, bc) = (a, c) + (a, b)c$

(4) if (Σ^*, V^*) is obtained from (Σ, V) by the fusion of $P, Q \in V$ and a^*/b^* denotes the image of a/b in $\Pi_1(\Sigma^*, V^*)$, then

$$(a^*, b^*) = (a, b)^* - (\delta_{\text{in}(a), P} a^{-1} - \delta_{\text{out}(a), P} 1_{P^*})(\delta_{\text{in}(b), Q} b - \delta_{\text{out}(b), Q} 1_{P^*}) + \\ + (\delta_{\text{in}(a), Q} a^{-1} - \delta_{\text{out}(a), Q} 1_{P^*})(\delta_{\text{in}(b), P} b - \delta_{\text{out}(b), P} 1_{P^*}).$$

It is the only natural map with these properties.

Proof. These properties are readily verified. Uniqueness can be seen by representing (Σ, V) as a fusion of a collection of discs, each with two marked points. \square

For $a, b \in \Pi_1(\Sigma, V)$ let us consider the $\mathfrak{g} \otimes \mathfrak{g}$ -valued function on $M_{\Sigma, V}$,

$$\pi(\text{hol}_a^* \theta^L, \text{hol}_b^* \theta^L).$$

These functions, in turn, specify π completely.

We can now state our version of the result of Massuyeau and Turaev [19].

Theorem 3. For any $\forall a, b \in \Pi_1(\Sigma, V)$ we have

$$(12) \quad \pi(\text{hol}_a^* \theta^L, \text{hol}_b^* \theta^L) = \frac{1}{2}(\mathbf{Ad}_{\text{hol}_{(a,b)}} \otimes 1) s.$$

Remark 4. Essentially the same formula was discovered independently by Xin Nie [20].

Proof of Theorem 3 and of Lemma 2. To prove both Theorem 3 and Lemma 2 (and thus finish the proof of Theorem 2) we need to check that

$$(13) \quad \pi_\Gamma(\text{hol}_a^* \theta^L, \text{hol}_b^* \theta^L) = \frac{1}{2}(\mathbf{Ad}_{\text{hol}_{(a,b)}} \otimes 1) s \quad (\forall a, b \in \Pi_1(\Sigma, V))$$

for any skeleton Γ of (Σ, V) .

Notice (via $\text{hol}_{bc}^* \theta^L = \text{hol}_c^* \theta^L + \mathbf{Ad}_{(\text{hol}_c)^{-1}} \text{hol}_b^* \theta^L$) that

$$\pi_\Gamma(\text{hol}_a^* \theta^L, \text{hol}_{bc}^* \theta^L) = \pi_\Gamma(\text{hol}_a^* \theta^L, \text{hol}_c^* \theta^L) + (1 \otimes \mathbf{Ad}_{(\text{hol}_c)^{-1}}) \pi_\Gamma(\text{hol}_a^* \theta^L, \text{hol}_b^* \theta^L).$$

As a result, if (13) is true for all a, b in a set of generators of $\Pi_1(\Sigma, V)$, it is then true (by Proposition 1 Part 3) for all elements of $\Pi_1(\Sigma, V)$.

Equation (13) is true if (Σ, V) is a disc with two marked points, as both sides of the equation vanish. The same is true for the disjoint union of a collection of such discs. As any (Σ, V, Γ) can be obtained from such a collection by a repeated fusion, it remains to check that (13) is preserved under fusion.

Suppose that (13) is satisfied for some (Σ, V) and its skeleton Γ . Let (Σ^*, V^*) be a fusion of (Σ, V) and let Γ^* be the image of Γ in Σ^* . Then π_{Γ^*} is obtained from π_Γ by the corresponding quasi-Poisson fusion. By Proposition 1 Part 4 we then get

$$\pi_{\Gamma^*}(\text{hol}_a^* \theta^L, \text{hol}_b^* \theta^L) = \frac{1}{2}(\mathbf{Ad}_{\text{hol}_{(a^*, b^*)}} \otimes 1) s.$$

In other words, (13) is satisfied also for $(\Sigma^*, V^*, \Gamma^*)$ for the elements of $\Pi_1(\Sigma^*, V^*)$ in the image of $\Pi_1(\Sigma, V)$. As the image generates $\Pi_1(\Sigma^*, V^*)$, we conclude that (13) is satisfied for $(\Sigma^*, V^*, \Gamma^*)$. \square

Remark 5. Theorem 3 can be used as an alternative definition of π . Properties 1–3 of the homotopy intersection form (cf. Proposition 1) mean that there is a unique G^V -invariant bivector field π satisfying (12). Property 4 means that π is compatible with fusion.

If $f : (\Sigma', V') \rightarrow (\Sigma, V)$ is an embedding then clearly

$$(f_* a, f_* b) = f_*(a, b) \quad (\forall a, b \in \Pi_1(\Sigma, V)).$$

From Theorem 3 we thus get the following result.

Corollary 1. *If $f : (\Sigma', V') \rightarrow (\Sigma, V)$ is an embedding then*

$$f^* : M_{\Sigma, V}(G) \rightarrow M_{\Sigma', V'}(G)$$

is a quasi-Poisson map, i.e. $f_ \pi_{\Sigma, V} = \pi_{\Sigma', V'}$.*

Let us now consider the special case $\Sigma' = \Sigma$, $V' \subset V$. Recall that if (M, ρ, π) is a $G \times H$ -quasi-Poisson manifold and if M/G is a manifold (e.g. if the action of G is free and proper) then π descends to a bivector field π' on M/G such that $p_* \pi = \pi'$, where $p : M \rightarrow M/G$ is the projection, and that M/G thus becomes a H -quasi-Poisson manifold, called the *quasi-Poisson reduction of M by G* . Using this terminology, Corollary 1 becomes

Corollary 2. *If $V' \subset V$ then $M_{\Sigma, V'}(G)$ is the quasi-Poisson reduction of $M_{\Sigma, V}(G)$ by $G^{V \setminus V'}$.*

Finally, again following Massuyeau and Turaev [19], we can define a moment map for the quasi-Poisson G^V -manifold $M_{\Sigma, V}(G)$. Let us orient $\partial\Sigma$ *against* the orientation induced from Σ . If we walk along $\partial\Sigma$ using this orientation, we get a permutation $\sigma : V \rightarrow V$. Let $\tau : G^V \rightarrow G^V$ be the automorphism

$$\tau(g)_v = g_{\sigma(v)} \quad (\forall g \in G^V, v \in V).$$

so that the τ -twisted action of $G^V \times G^V$ on G^V is

$$(14) \quad ((g_1, g_2) \cdot g)_v = (g_1)_{\sigma(v)} g_v (g_2)_v^{-1}$$

(cf. Eq. (4)). For every $v \in V$ let $\mu_v : M_{\Sigma, V}(G) \rightarrow G$ be

$$\mu_v = \text{hol}_{a_v}$$

where a_v is the boundary arc from v to $\sigma(v)$. Let us combine the maps μ_v to a single map $\mu : M_{\Sigma, V}(G) \rightarrow G^V$.

Theorem 4. *The map $\mu : M_{\Sigma, V}(G) \rightarrow G^V$ is a τ -twisted moment map.*

Proof. The equivariance of μ is obvious. If $v \in V$ and $b \in \Pi_1(\Sigma, V)$ then by the definition of (a_v, b) we have

$$(a_v, b) = -\delta_{\text{out}(b), v} 1_v - \delta_{\text{out}(b), \sigma(v)} a_v^{-1} + \delta_{\text{in}(b), v} b + \delta_{\text{in}(b), \sigma(v)} a_v^{-1} b.$$

Theorem 3 then implies that μ is a moment map, as the 1-forms $\text{hol}_b^* \theta^L$ span the cotangent bundle of $M_{\Sigma, V}(G)$. \square

Remark 6. An alternative way of proving Theorem 4 is to verify it for the case of a disc with two marked points on the boundary, and then to use fusion.

6. SURFACES WITH BOUNDARY DATA

For every point $v \in V$ let us choose a coisotropic subgroup $C_v \subseteq G$.⁶ Let $C = \prod_{v \in V} C_v \subseteq G^V$. Suppose the C -orbits in $M_{\Sigma, V}(G)$ form a regular foliation, and consider the orbit space

$$M_{\Sigma, V}(G, (C_v)_{v \in V}) := M_{\Sigma, V}(G)/C.$$

Geometrically, $M_{\Sigma, V}(G, (C_v)_{v \in V})$ is the moduli space of flat connections on principal G -bundles $P \rightarrow \Sigma$ with a reduction to C_v over v for every $v \in V$.

Since $C \subseteq G^V$ is coisotropic, by Theorem 1.A we know that $M_{\Sigma, V}(G, (C_v)_{v \in V})$ is Poisson. More generally, if $V' \subseteq V$ then Theorem 1.C implies that

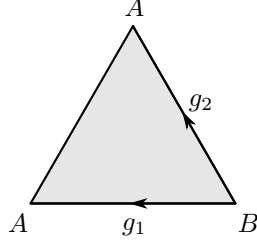
$$M_{\Sigma, V}(G) / \prod_{v \in V'} C_v$$

⁶If G is semisimple then any parabolic subgroup is coisotropic; if G is simple then these are all the coisotropic subgroups

is a quasi-Poisson $G^{V \setminus V'}$ -manifold.

Example 4. Let Σ be a triangle and V the set of its vertices. Suppose that $s \in S^2\mathfrak{g}$ is non-degenerate, and that $(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ is a Manin triple. Let $A, B \subset G$ be the corresponding subgroups, and let us suppose that $A \cap B = \{1\}$ and $AB = G$.

Let us choose the subgroup, C_v , at two of the vertices to be A and the remaining vertex to be B , as in the picture below



Now the holonomies $g_1, g_2 \in G$ along the edges pictured above identify $M_{\Sigma, V}(G)$ with $G \times G$, where the bivector field was described in Eq. (9). The diffeomorphism

$$((a, b) \rightarrow ab) : A \times B \rightarrow G$$

identifies $M_{\Sigma, V}$ with $A \times B \times A \times B$, and the action of $C = A \times B \times A$ on $M_{\Sigma, V}(G)$ becomes

$$(a, b, a') \cdot (a_1, b_1, a_2, b_2) = (aa_1, b_1b^{-1}, a'a_2, b_2b^{-1}).$$

Thus the map $(a_1, b_1, a_2, b_2) \rightarrow b_1b_2^{-1}$ identifies $M_{\Sigma, V}(G)/C$ with the Lie group B .

A comparison shows that the Poisson structure on $M_{\Sigma, V}(G)/C \cong B$ is the Poisson-Lie structure on B described in [17].

Example 5. If Σ, V is a disc with two marked points and $C \subseteq G$ is a coisotropic subgroup which we embed as a subgroup of the second factor of $G \times G$, then $M_{\Sigma, V}(G)/C = G/C$ is a quasi-Poisson G -manifold, with $\pi = 0$.

Since, according to Theorem 4, the holonomies along the boundary arcs define a moment map $\mu : M_{\Sigma, V}(G) \rightarrow G^V$, we can apply the moment map reduction (Theorem 1.B) to get Poisson submanifolds of $M_{\Sigma, V}(G, (C_v)_{v \in V}) = M_{\Sigma, V}(G)/C$.⁷ Let us give a geometric description of these Poisson submanifolds, by constructing \hat{M} , $\hat{\mu}$ and \hat{C} , as in Section 3.

Recall that $\sigma : V \rightarrow V$ is the permutation obtained by walking along $\partial\Sigma$ against the orientation induced from Σ , and that a_v is the boundary arc from v to $\sigma(v)$.

Let us first describe the group \hat{C} . Let $\mathfrak{c}_v^\perp = s^\# \text{ann}(\mathfrak{c}_v)$; it is an ideal in \mathfrak{c}_v . Let $C_v^\perp \subseteq C_v$ denote the corresponding (connected) Lie group.⁸ Let

$$\hat{C}_v = \{(g_1, g_2) \in C_v \times C_v; g_1^{-1}g_2 \in C_v^\perp\} \subset G \times G.$$

Then

$$\hat{C} = \prod \hat{C}_v \subset G^V \times G^V.$$

Let us give a geometrical description for the manifold \hat{M} (see Eq. (6)) and of the map $\hat{\mu} : \hat{M} \rightarrow G^V$ in the case of $M = M_{\Sigma, V}(G)$. First, let $\hat{\Sigma}$ be the surface obtained from Σ by blowing up at each point $v \in V$, as in Fig. 5. We let w_v denote the exceptional divisor obtained by blowing up at v . With a slight abuse of notation, we label the initial and end points of the segment w_v by \bar{v} and v , respectively. We

⁷If $s \in S^2\mathfrak{g}$ is nondegenerate and if every component of $\partial\Sigma$ contains a marked point then these submanifolds are in fact the symplectic leaves.

⁸If G is simple, so that C_v is parabolic, then $C_v^\perp \subseteq C_v$ is the nilpotent radical

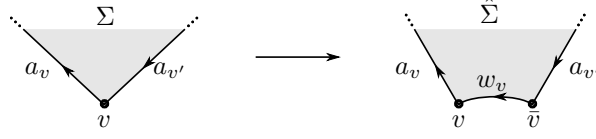


FIGURE 5. $\hat{\Sigma}$ is obtained from Σ by blowing up at each point $v \in V$. We denote the exceptional divisor (a segment on the boundary of Σ) by w_v , its initial point by \bar{v} and its endpoint by v . Note that $v = \sigma(v')$, and the orientation of the arcs is opposite to the induced boundary orientation.

let W denote the set of w_v 's, and we let \bar{V} and V denote the set of initial and end points of the w_v 's. Thus $(\hat{\Sigma}, \bar{V} \cup V)$ is a marked surface.

Then

$$\hat{M} = \{f \in M_{\hat{\Sigma}, \bar{V} \cup V}(G); f(w_v) \in C_v^\perp (\forall v \in V)\}.$$

The group $G^V \times G^V \cong G^{\bar{V} \cup V}$ acts naturally on $M_{\hat{\Sigma}, \bar{V} \cup V}(G)$ (cf. Eq. (8)) and the subgroup \hat{C} preserves $\hat{M} \subseteq M_{\hat{\Sigma}, \bar{V} \cup V}(G)$. Under this action, an element of \hat{C} is identified with $(g_{\bar{v}}, g_v)_{v \in V}$ such that $g_v g_{\bar{v}}^{-1} \in C_v^\perp$.

Suppose we now choose an element $h = (h_v)_{v \in V} \in G^V$. Recall from Eq. (14) that the action of $\hat{C} \subseteq G^{\bar{V} \cup V}$ on G^V is by

$$(g \cdot h)_v = g_{\sigma(v)} h_v g_v^{-1}.$$

Let $\mathcal{O} \subseteq G^V$ denote the \hat{C} -orbit containing $h \in G^V$. Using Eq. (7) we get

$$\mu^{-1}(\mathcal{O})/C \cong \hat{\mu}^{-1}(\mathcal{O})/\hat{C} \cong \{f \in \hat{M}; f(a_v) = h_v (\forall v \in V)\}/\text{Stab}(h).$$

Thus, we have proven

Theorem 5. *For any $h = (h_v)_{v \in V} \in G^V$ the space*

$$\{f \in M_{\hat{\Sigma}, \bar{V} \cup V}(G); f(w_v) \in C_v^\perp, f(a_v) = h_v (\forall v \in V)\}/\text{Stab}(h),$$

(where $\text{Stab}(h)$ is the stabilizer of h in \hat{C}) is naturally isomorphic to a Poisson submanifold of $M_{\Sigma, V}(G)/C$.

Theorem 5 is particularly interesting in the case when $h = \mathbf{1}$ is the unit. Since this constrains the holonomies along the paths a_v to be trivial, we can contract these paths to points.

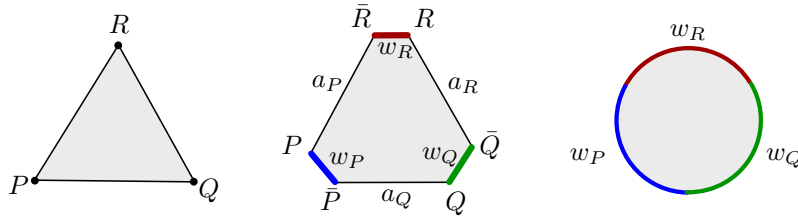
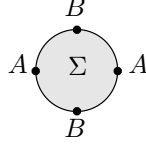


FIGURE 6. Blow up at the marked points, followed by contraction of a_v 's

Example 6. [[23, 25, 24]] Once again, let $(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ be a Manin triple, let Σ, V be a square, and let us choose the subgroups C_v as in the picture,



where $A, B \subset G$ are Lie groups integrating \mathfrak{a} and \mathfrak{b} . If we suppose $AB = G$ and $A \cap B = \{1\}$, then the action of \hat{C} on G^V is free and transitive. We can thus set $h = 1$, and get the constraint on the holonomies as in Fig. 7.

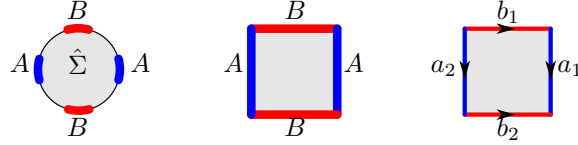


FIGURE 7. Setting $h = 1$ is effectively the same as contracting the uncolored portions of the boundary in the first picture; this results in the second picture. We label the holonomies as in the third picture.

In this case $\text{Stab}(h) = 1$ and $M_{\Sigma,V}(G)/C$ is

$$\{(a_1, b_1, a_2, b_2) \in A \times B \times A \times B; a_1 b_1 = b_2 a_2\},$$

where the holonomies are as pictured in Fig. 7. We can identify the moduli space with G via

$$g = a_1 b_1 = b_2 a_2.$$

The resulting Poisson structure on G is the so-called Heisenberg double. In another terminology, $M_{\Sigma,V}(G)/C$ is the Lu-Weinstein double symplectic groupoid (cf. [17]). This moduli space was first studied in the work of the second author [23, 25, 24].

Description in terms of flat connections. We now give an alternative description of our Poisson submanifolds of $M_{\Sigma,V}/C$ in terms of flat connections.

Definition 4. Boundary data for a principal G -bundle $P \rightarrow \Sigma$ is $P|_{\partial\Sigma} \rightarrow \partial\Sigma$ together with a choice of a reduction $Q_v \rightarrow a_v$ of $P|_{a_v} \rightarrow a_v$ to $C_v \subseteq G$, and of a flat connection on the principal C_v/C_v^\perp -bundle Q_v/C_v^\perp , for every $v \in V$.

We shall say that a flat connection on $P \rightarrow \Sigma$ is *compatible* with a given choice of boundary data if the flat connection restricts to each Q_v and if it gives rise to the pre-assigned flat connection on Q_v/C_v^\perp . Notice that for any choice of a flat connection on P , any reduction of P to C_v over v (for all $v \in V$) can be extended in a unique way to define compatible boundary data (using parallel transport). We can therefore equate $M_{\Sigma,V}(G, (C_v)_{v \in V})$ with the moduli space of triples

$$(P \rightarrow \Sigma, \text{boundary data, a compatible flat connection}).$$

Theorem 5'. The subset of $M_{\Sigma,V}(G, (C_v)_{v \in V})$ corresponding to a given isomorphism class of boundary data is a Poisson submanifold, provided both $M_{\Sigma,V}(G)/C$ and the given subset are manifolds.

To prove Theorem 5' we can use a slightly different (but equivalent) description of \hat{M} . Let us consider principal bundles $P \rightarrow \Sigma$ with a flat connection and compatible boundary data, together with a trivialization of $Q_v \rightarrow a_v$ over the endpoints v and $\sigma(v)$ of a_v , compatible with the flat connection on $Q_v/C_v^\perp \rightarrow a_v$. Isomorphism classes of such objects are the points of \hat{M} . \hat{C} acts on \hat{M} by changing the trivializations.

7. SURFACES WITH DOMAIN WALLS

Consider a finite family (Σ_d, V_d) indexed by a set, **Dom**. For each $d \in \mathbf{Dom}$ choose a Lie group G_d and an element $s_d \in (S^2 \mathfrak{g}_d)^{G_d}$. Then

$$M = \prod_d M_{\Sigma_d, V_d}(G_d)$$

is a quasi-Poisson G -manifold, where

$$G = \prod_d G_d^{V_d}.$$

Let us now construct a reducing subgroup of G in the following way. First, we split the set

$$V = \sqcup_d V_d$$

of all the marked points into (a disjoint union of) pairs and singletons. For every singleton $v \in V_d$ we choose a coisotropic subgroup

$$C_v \subseteq G_d.$$

For every pair $v \in V_d, v' \in V_{d'}$ we choose a subgroup

$$C_{v,v'} \subseteq G_d \times G_{d'}$$

which is either coisotropic w.r.t. $s_d + s_{d'}$, in which case we shall call the pair v, v' *anti-oriented*, or coisotropic w.r.t. $s_d - s_{d'}$, in which case we shall call the pair *oriented*. The product

$$C = \prod_v C_v \times \prod_{(v,v')} C_{v,v'} \subseteq \prod_d G_d^{V_d} = G$$

is a reducing subgroup of G . Indeed, it is coisotropic after we change the sign of $s_{d'}$ at every oriented pair of vertices $(v, v') \in V_d \times V_{d'}$, as in Remark 1.

As a result (by Theorem 1.A),

$$M/C$$

is a Poisson manifold (provided the quotient space is a manifold).

Let us now construct \hat{M} corresponding to $M = \prod_d M_{\Sigma_d, V_d}(G_d)$, the action of \hat{C} on \hat{M} , and the map $\hat{\mu} : \hat{M} \rightarrow G$. Once again, we construct $\hat{\Sigma}_d$ by blowing up Σ_d at every $v \in V_d$. We let $w_v \subset \hat{\Sigma}_d$ denote the preimage of $v \in V_d$ and denote the boundary points of w_v by \bar{v} and v (see Fig. 5). Let us now sew w_v with $w_{v'}$ for every v, v' forming a pair; if the pair is oriented then the sewing is done so that the orientations of $\hat{\Sigma}_d$ and $\hat{\Sigma}_{d'}$ agree on both sides of the sewn edge (i.e. v is sewed to \bar{v}' and v' to \bar{v}), if the pair is anti-oriented, the sewing is done so that the orientation of $\hat{\Sigma}_d$ and $\hat{\Sigma}_{d'}$ disagree. Following Wehrheim and Woodward [28, 29] we shall call the resulting surface Σ a *quilted surface*, and the images of the w_v 's in Σ the *domain walls*.

To summarize, for every domain of the quilted surface we have a Lie group G_d ; for every domain wall we have a coisotropic subgroup. The set of domain walls will be denoted **Wall**; we shall orient every $\mathbf{w} \in \mathbf{Wall}$ in an arbitrary way. For every $\mathbf{w} \in \mathbf{Wall}$ let $C_{\mathbf{w}}$ be the corresponding C_v or $C_{v,v'}$.

We can now describe \hat{M} in terms of the quilted surface. An element $f \in \hat{M}$ is a collection of elements $f_d \in M_{\hat{\Sigma}_d, \bar{V}_d \cup V_d}(G_d)$ ($d \in \mathbf{Dom}$), satisfying a condition for every domain wall:

$$f(\mathbf{w}) \in C_{\mathbf{w}}^\perp \quad (\forall \mathbf{w} \in \mathbf{Wall}).$$

Here $f(\mathbf{w})$ is the holonomy (if \mathbf{w} is on the boundary of Σ) or the pair of holonomies (if \mathbf{w} is inside Σ) along \mathbf{w} .

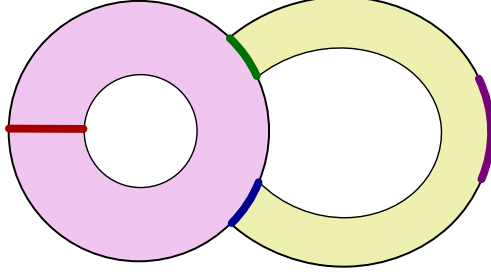


FIGURE 8. A quilted surface with 2 domains and 4 domain walls

The groups $C_{\mathbf{w}}^{\perp}$ are defined as follows. For every singleton $v \in V_d$ let

$$\mathbf{c}_v^{\perp} = s_d^{\#} \text{ann}(\mathbf{c}_v),$$

for every anti-oriented pair $v \in V_d, v' \in V_{d'}$ let

$$\mathbf{c}_{v,v'}^{\perp} = (s_d + s_{d'})^{\#} \text{ann}(\mathbf{c}_{v,v'}),$$

and for every oriented pair $v \in V_d, v' \in V_{d'}$ let

$$\mathbf{c}_{v,v'}^{\perp} = (s_d - s_{d'})^{\#} \text{ann}(\mathbf{c}_{v,v'}).$$

Then $C_v^{\perp} \subseteq C_v$ and $C_{v,v'}^{\perp} \subseteq C_{v,v'}$ are the corresponding normal subgroups.

For every $\mathbf{w} \in \mathbf{Wall}$ let

$$\hat{C}_{\mathbf{w}} = \{(g_{\bar{\mathbf{w}}}, g_{\mathbf{w}}) \in C_{\mathbf{w}} \times C_{\mathbf{w}}; g_{\mathbf{w}} g_{\bar{\mathbf{w}}}^{-1} \in C_{\mathbf{w}}^{\perp}\} \subseteq C_{\mathbf{w}} \times C_{\mathbf{w}}.$$

Then $\hat{C} = \prod_{\mathbf{w}} \hat{C}_{\mathbf{w}}$. Every $\hat{C}_{\mathbf{w}}$ acts on \hat{M} by acting at the endpoints of \mathbf{w} , this defines the action of \hat{C} on \hat{M} .

Finally, the map

$$\hat{\mu} : \hat{M} \rightarrow G$$

is the collection of holonomies along the arcs a_v , $v \in V$ (i.e. along the boundary arcs of Σ which are not domain walls).

The isomorphism

$$\hat{M}/\hat{C} \cong M/C$$

can be seen via the embedding $M \hookrightarrow \hat{M}$ given by the constraint $f(\mathbf{w}) = \mathbf{1}$ for all $\mathbf{w} \in \mathbf{Wall}$ (effectively contracting every \mathbf{w} to a point). We write

$$(15) \quad \mathcal{M}_{\Sigma} := \hat{M}/\hat{C}$$

Theorem 1.B and Eq. (7) now give the following result.

Theorem 6.A. *Let $h \in G^V$. If both the quotient space*

$$(16) \quad N := \{f \in \hat{M}; f(a_v) = h_v\} / \text{Stab}(h)$$

and M/C are manifolds, then N is naturally isomorphic to a Poisson submanifold of M/C .

The theorem is particularly interesting when $h = \mathbf{1}$, as then we can contract the arcs a_v . More generally we might want to only contract a subset of the arcs a_v . For instance, suppose we want to contract those arcs a_v with $v \in V_{triv} \subseteq \cup_{d \in \mathbf{Dom}} V_d$. We can do this as follows: Let

$$H = \{h \in \prod_{d \in \mathbf{Dom}} G_d^{V_d}; h_v = \mathbf{1} \text{ for all } v \in V_{triv}\},$$

and

$$\hat{H} = \{g \in \prod_{d \in \mathbf{Dom}} G_d^{\bar{V}_d \cup V_d}; g_v = g_{\sigma(v)} \text{ for all } v \in V_{triv}\}.$$

Then for any $g \in \prod_d G_d^{\bar{V}_d \cup V_d}$, and $h \in H$,

$$(17) \quad g \cdot h \in H \Rightarrow g \in \hat{H}.$$

Suppose $K = \hat{C} \cdot H$ is a submanifold of $\prod_d G_d^{V_d}$ transverse to μ . Then Eq. (17) implies

$$\mu^{-1}(K)/C \cong \hat{\mu}^{-1}(\hat{C} \cdot H)/\hat{C} \cong \hat{\mu}^{-1}(H)/(\hat{C} \cap \hat{H})$$

is a Poisson submanifold of M/C . Thus we have shown:

Theorem 6.B. *The moduli space*

$$\{f \in \hat{M}; f(a_v) = \mathbf{1} \text{ for all } v \in V_{triv}\}/(\hat{C} \cap \hat{H})$$

is naturally isomorphic to a submanifold of M/C .

In the sequel, we will refer to

$$\hat{C} \cap \hat{H} = \left\{ g \in \prod_d G_d^{\bar{V}_d \cup V_d} \left| \begin{array}{l} g_{\mathbf{w}}, g_{\mathbf{w}} \in C_{\mathbf{w}} \\ g_{\mathbf{w}} g_{\mathbf{w}}^{-1} \in C_{\mathbf{w}}^{\perp} \\ g_v = g_{\sigma(v)} \end{array} \right. \right\}$$

as the *group of residual gauge transformations*. Here $g_{\mathbf{w}} := g_v$ when the domain wall \mathbf{w} was obtained from the singleton $v \in V$, and $g_{\mathbf{w}} := (g_v, g_{v'})$ when the domain wall \mathbf{w} was obtained from the pair $v, v' \in V$.

We leave the reformulation of Theorem 6 in terms of flat connections (in the spirit of Theorem 5') to the reader.

Quilted surfaces with residual marked points. In this section, we describe a slight generalization of the above theory which allows us to leave the marked points on certain boundary components of our domains Σ_d ($d \in \mathbf{Dom}$) unreduced (or unsewn). The resulting moduli spaces are quasi-Poisson rather than Poisson. Suppose that we decompose the set $B := \sqcup_d \partial \Sigma_d$ of all boundary components into two closed subsets $B = B^{sew} \sqcup B^{res}$. We will leave those marked points $V^{res} := V \cap B^{res}$ in B^{res} unreduced (here $V = \sqcup_d V_d$). As in the previous section we split the complementary set $V^{sew} := V \cap B^{sew}$ of marked points into a disjoint union of pairs and singletons. For every singleton $v \in V_d^{sew} := V_d \cap V^{sew}$ we choose a coisotropic subgroup $C_v \subseteq G_d$ and for every pair $v \in V_d^{sew}, v' \in V_{d'}^{sew}$ we choose a subgroup $C_{v,v'} \subseteq G_d \times G_{d'}$ which is coisotropic with respect to $s_d \pm s_{d'}$. The product

$$C = \prod_v C_v \times \prod_{(v,v')} C_{v,v'}$$

is a reducing subgroup of $G := \prod_d G_d^{V_d^{sew}}$.

With $H := \prod_d G_d^{V_d \cap V^{res}}$, Theorem 1.C implies that whenever the C -orbits of

$$M = \prod_d M_{\Sigma_d, V_d}(G_d)$$

form a regular foliation, then

$$M/C$$

is a quasi-Poisson H -manifold. As before, we construct $\hat{\Sigma}_d$ by blowing up Σ_d at every $v \in V_d^{sew}$, we let $w_v \subset \hat{\Sigma}_d$ denote the preimage of v and we let \bar{v} and v denote the boundary points of w_v . We form the quilted surface Σ by sewing w_v and $w_{v'}$ for every pair v, v' (orienting them appropriately, as explained in the previous section).

As in the previous section, we let \hat{M} denote the subset of elements

$$f \in \prod_d M_{\hat{\Sigma}_d, V_d^{res} \cup \bar{V}_d^{sew} \cup V_d^{sew}}(G_d)$$

which satisfy the additional condition that

$$f(\mathbf{w}) \in C_{\mathbf{w}}^{\perp}, \quad (\forall \mathbf{w} \in \mathbf{Wall}),$$

where \mathbf{Wall} denotes the set of domain walls (the images of the w_v 's in Σ).

The same considerations as in the previous section yield the following result:

Theorem 6.C. *The moduli space*

$$\{f \in \hat{M}; f(a_v) = \mathbf{1} \text{ for all } v \in V^{sew}\} / \text{Stab}(\mathbf{1})$$

is naturally isomorphic to a quasi-Poisson submanifold of M/C , where

$$\text{Stab}(\mathbf{1}) = \{\hat{c} \in \hat{C}; \hat{c} \cdot \mathbf{1} = \mathbf{1}\}.$$

As before, constraining the holonomy along the arc a_v to equal $\mathbf{1}$ is equivalent to contracting that arc.

8. SPIN NETWORKS

In this section, we reinterpret the quasi-Poisson structure on marked surfaces as well as the Poisson structure on quilted surfaces in terms of spin networks.

Remark 7. Spin networks were first introduced by Penrose [21] (see also [6]). Poisson brackets of spin networks were studied by Roche and Szenes [22], following work by Goldman [13, 12], Andersen, Mattes and Reshetikhin [4].

8.1. Spin networks on a marked surface. Let (Σ, V) denote a marked surface, as in Section 4.

Definition 5. *A graph diagram in (Σ, V) consists of the following data:*

- *a directed graph, Γ with edges E_{Γ} and vertices V_{Γ} ,*
- *a subset of vertices $V_{\Gamma}^{anch} \subseteq V_{\Gamma}$, called ‘anchor points’, and*
- *a smooth map $\iota : \Gamma \rightarrow \Sigma$ such that*

$$\iota^{-1}(V) = V_{\Gamma}^{anch}.$$

Often we will abuse notation and denote the graph diagram simply by Γ .

A morphism between graph diagrams $\iota : \Gamma \rightarrow \Sigma$ and $\iota' : \Gamma' \rightarrow \Sigma$ is a map $\mu : \Gamma \rightarrow \Gamma'$ such that $\iota = \iota' \circ \mu$ and $\mu(V_{\Gamma}) \subseteq V_{\Gamma'}$.

A homotopy between graph diagrams $\iota_0 : \Gamma \rightarrow \Sigma$ and $\iota_1 : \Gamma \rightarrow \Sigma$ is one parameter family of graph diagrams $\iota_t : \Gamma \rightarrow \Sigma$, $t \in [0, 1]$.

Suppose Γ is a graph diagram in (Σ, V) . The Lie group $G^{V_{\Gamma}}$ acts on $G^{E_{\Gamma}}$ by

$$(g \cdot g')_e = g_{\text{in}(e)} g'_e g_{\text{out}(e)}^{-1},$$

where $g \in G^{V_{\Gamma}}$, $g' \in G^{E_{\Gamma}}$, and $e \in E_{\Gamma}$. We define

$$\mathcal{A}_{\Gamma}(G) := C^{\infty}(G^{E_{\Gamma}})^{G^{V_{\Gamma}^{int}}},$$

where $V_{\Gamma}^{int} = V_{\Gamma} \setminus V_{\Gamma}^{anch}$. Note that there is a residual action of $G^{V_{\Gamma}^{anch}}$ on $\mathcal{A}_{\Gamma}(G)$. For any $v \in V_{\Gamma}^{anch}$, we let $\rho_v : G \times \mathcal{A}_{\Gamma}(G) \rightarrow \mathcal{A}_{\Gamma}(G)$ denote the action of the v^{th} -factor.

Remark 8. Since Γ is a graph, $G^{E_{\Gamma}} = \text{Hom}(\Pi_1(\Gamma, V_{\Gamma}), G)$. Therefore

$$\mathcal{A}_{\Gamma}(G) = C^{\infty}\left(\text{Hom}(\Pi_1(\Gamma, V_{\Gamma}), G)\right)^{G^{V_{\Gamma}^{int}}}.$$

By functoriality, a morphism $\mu : \Gamma \rightarrow \Gamma'$ of graph diagrams defines a morphism

$$\mu^! : \text{Hom}(\Pi_1(\Gamma', V_{\Gamma'}), G) \rightarrow \text{Hom}(\Pi_1(\Gamma, V_{\Gamma}), G),$$

which is equivariant with respect to the map

$$\mu_V^! : G^{V_{\Gamma'}} \rightarrow G^{V_{\Gamma}}$$

defined by $\mu^!(g)_v = g_{\mu(v)}$, for any $g \in G^{V_{\Gamma'}}$. Since $\mu_V^!$ maps the normal subgroup $G^{V_{\Gamma'}^{int}}$ into $G^{V_{\Gamma}^{int}}$, pull-back along $\mu^!$ defines an equivariant morphism of algebras $\mu_* : \mathcal{A}_{\Gamma} \rightarrow \mathcal{A}_{\Gamma'}$. We may summarize this as:

Lemma 3. *The assignment $(\iota : \Gamma \rightarrow \Sigma) \mapsto \mathcal{A}_{\Gamma}$ is a functor from graph diagrams to algebras endowed with an action of G^V , where $g \in G^V$ acts via*

$$g \cdot f = \left(\prod_{v \in V_{\Gamma}^{anch}} \rho_v(g_{\iota(v)}) \right) f.$$

Definition 6. *A spin network in (Σ, V) is a pair (Γ, f) , where $\Gamma \rightarrow \Sigma$ is a graph diagram, and $f \in \mathcal{A}_{\Gamma}(G)$.*

We say that spin networks (Γ_0, f_0) and (Γ_1, f_1) are homotopic if the underlying graph diagrams are homotopic and $f_0 = f_1 \in \mathcal{A}_{\Gamma_0} \equiv \mathcal{A}_{\Gamma_1}$ (to identify the algebras, we use the fact that the definition of \mathcal{A}_{Γ} only depends on the underlying graph, and not the map $\Gamma \rightarrow \Sigma$).

A morphism of spin networks $\mu : (\Gamma, f) \rightarrow (\Gamma', f')$ is a morphism of graph diagrams $\mu : \Gamma \rightarrow \Gamma'$ such that $f' = \mu_ f$.*

We consider two spin networks to be equivalent if they are related by a chain of homotopies and morphisms (or the formal inverses of morphisms). For a spin network (Γ, f) , we let $[\Gamma, f]$ denote the corresponding equivalence class. Define $\text{SpinNet}_{(\Sigma, V)}(G)$ to be the set of equivalence classes of spin networks.

Lemma 4. *$\text{SpinNet}_{(\Sigma, V)}(G)$ is a G^V -algebra where scalar multiplication, addition, and multiplication are defined as*

$$(18a) \quad \lambda \cdot [\Gamma, f] = [\Gamma, \lambda f]$$

$$(18b) \quad [\Gamma, f] + [\Gamma', f'] = [\Gamma \cup \Gamma', f \oplus f']$$

$$(18c) \quad [\Gamma, f] \cdot [\Gamma', f'] = [\Gamma \cup \Gamma', f \otimes f']$$

Lemma 4 will follow from Proposition 2.

8.2. Spin networks and functions on the moduli space. Suppose (Γ, f) is a spin network in (Σ, V) . Then we may push f along the map $\iota : \Gamma \rightarrow \Sigma$ to define a function $\text{ev}(\Gamma, f)$ on the moduli space $M_{\Sigma, V}(G)$, as we shall now describe.

For a finite set of points $X = \{x_i\} \in \Sigma \setminus V$, we let

$$M_{\Sigma, V, X}(G) := \text{Hom}(\Pi_1(\Sigma, V \cup X), G)$$

Notice that

$$M_{\Sigma, V}(G) = M_{\Sigma, V, X}(G)/G^X.$$

The map

$$\iota : (\Gamma, V_{\Gamma}) \rightarrow (\Sigma, V \cup V_{\Gamma}^{int})$$

yields a $G^V \times G^{V_{\Gamma}^{int}}$ -equivariant map

$$(19) \quad \iota^! : M_{\Sigma, V, V_{\Gamma}^{int}}(G) \rightarrow \text{Hom}(\Pi_1(\Gamma, V_{\Gamma}), G).$$

Therefore, the function

$$\text{ev}(\Gamma, f) := \iota^{!*} f \in C^{\infty}(M_{\Sigma, V, V_{\Gamma}^{int}}(G))$$

is $G^{V_{\Gamma}^{int}}$ -invariant. Hence it descends to define a function on the moduli space

$$M_{\Sigma,V}(G) \cong M_{\Sigma,V,V_{\Gamma}^{int}}(G)/G^{V_{\Gamma}^{int}}.$$

Moreover, the map

$$\text{ev}(\Gamma, \cdot) : \mathcal{A}_{\Gamma}(G) \rightarrow C^{\infty}(M_{\Sigma,V}(G))$$

is G^V -equivariant.

Notice that $\text{ev}(\Gamma_0, f_0) = \text{ev}(\Gamma_1, f_1)$ whenever (Γ_0, f_0) and (Γ_1, f_1) are homotopic. Moreover, if

$$\mu : (\iota : \Gamma \rightarrow \Sigma) \rightarrow (\iota' : \Gamma' \rightarrow \Sigma)$$

is a morphism of graph diagrams, then following diagram of equivariant morphisms commute

$$\begin{array}{ccc} M_{\Sigma,V,V_{\Gamma'}^{int}}(G) & & \\ \downarrow \iota'^! & \searrow \iota^! & \\ \text{Hom}(\Pi_1(\Gamma', V_{\Gamma'}), G) & \xrightarrow{\mu^!} & \text{Hom}(\Pi_1(\Gamma, V_{\Gamma}), G) \end{array}$$

Therefore, $\iota'^!(f) = \iota'^! \circ \mu_*(f)$ for any $f \in \mathcal{A}_{\Gamma}$, i.e. $\text{ev}(\Gamma', \mu_*f) = \text{ev}(\Gamma, f)$. So ev descends to a map on the set of equivalence classes of spin networks, $\text{SpinNet}_{\Sigma,V}(G)$.

Proposition 2. *The map*

$$\text{ev} : \text{SpinNet}_{\Sigma,V}(G) \rightarrow C^{\infty}(M_{\Sigma,V}(G))$$

is an isomorphism of G^V -algebras.

Proof. With only a slight modification, the statement follows from the proofs of the corresponding statements for (unmarked) surfaces in [6, 22], but we outline it here for completeness.

Let $\widetilde{\text{SpinNet}}_{\Sigma,V}(G)$ be the set of all spin networks in (Σ, V) (i.e. we do not identify equivalent spin networks). We may define the operations in Eqs. 18 directly on $\widetilde{\text{SpinNet}}_{\Sigma,V}(G)$ (we do not claim that they satisfy the axioms of an algebra on this set). Since $(\Gamma, f) \rightarrow \text{ev}(\Gamma, f)$ is induced by the map of spaces (19), it follows that ev intertwines the operations of scalar multiplication, addition, and multiplication.

Next we show that $\text{ev} : \widetilde{\text{SpinNet}}_{\Sigma,V}(G) \rightarrow C^{\infty}(M_{\Sigma,V}(G))$ is surjective. Let $\iota_{skel} : \Gamma_{skel} \rightarrow \Sigma$ be an embedded graph diagram with a single anchor point at every marked point, for which there exists a deformation retract $r : \Sigma \rightarrow \iota_{skel}(\Gamma_{skel})$. Then Eq. (19),

$$\text{Hom}(\Pi_1(\Sigma, V \cup V_{\Gamma}^{int}), G) \rightarrow \text{Hom}(\Pi_1(\Gamma, V_{\Gamma}), G)$$

is a diffeomorphism. It follows that $\text{ev} : \mathcal{A}_{\Gamma_{skel}} \rightarrow C^{\infty}(M_{\Sigma,V}(G))$ is an isomorphism.

It remains to show that $\text{ev}(\Gamma, f) = \text{ev}(\Gamma', f')$ only if the two spin networks are equivalent. We may assume that there exist maps $\mu : \Gamma \rightarrow \Gamma_{skel}$ (just compose the map $\Gamma \rightarrow \Sigma$ with the retract $\Sigma \rightarrow \Gamma_{skel}$) and likewise $\mu' : \Gamma' \rightarrow \Gamma_{skel}$. Since $\text{ev} : \mathcal{A}_{\Gamma_{skel}} \rightarrow C^{\infty}(M_{\Sigma,V}(G))$ is an isomorphism, $\text{ev}(\Gamma, f) = \text{ev}(\Gamma', f')$ only if $\mu_*f = \mu'_*f'$, i.e. (Γ, f) and (Γ', f') are equivalent. \square

8.3. The quasi-Poisson bracket on $\text{SpinNet}_{\Sigma,V}(G)$. Suppose that $\iota : \Gamma \rightarrow \Sigma$ and $\iota' : \Gamma' \rightarrow \Sigma$ are graph diagrams such that the maps ι and ι' are transverse to each other.⁹ Let $(\Gamma \times_{\Sigma} \Gamma')^{anch} = (V_{\Gamma}^{anch}) \times_{\Sigma} (V_{\Gamma'}^{anch})$ and

$$(\Gamma \times_{\Sigma} \Gamma')^{int} = (\Gamma \times_{\Sigma} \Gamma') \setminus (\Gamma \times_{\Sigma} \Gamma')^{anch}.$$

⁹That is, the restriction of ι and ι' to any two edges are transverse, and $\iota(V_{\Gamma}^{int}) \cap \iota'(V_{\Gamma'}^{int}) = \emptyset$.

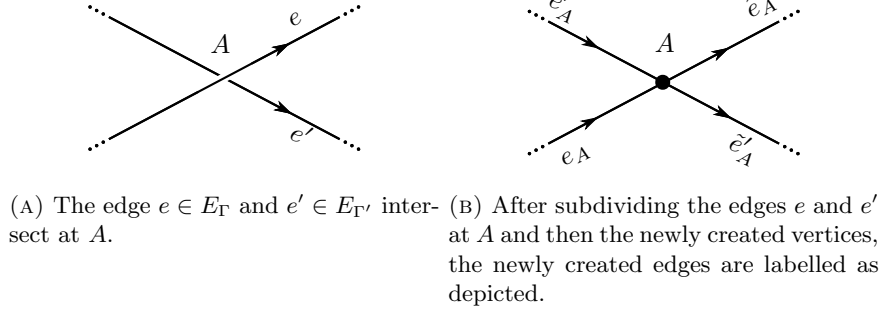


FIGURE 9

Given $A \in (\Gamma \times_\Sigma \Gamma')^{int}$, let $e \in E_\Gamma$ and $e' \in E_{\Gamma'}$ be the two edges intersecting at A . Let $\Gamma \cup_A \Gamma'$ be the graph obtained from $\Gamma \cup \Gamma'$ by subdividing the edges e and e' at A and then merging the newly created vertices. It is clear that

$$\iota \cup \iota' : \Gamma \cup_A \Gamma' \rightarrow \Sigma$$

is a graph diagram. We let e_A and e'_A denote the newly created edges running from $\text{out}(e)$ and $\text{out}(e')$ (respectively) to A and \tilde{e}_A and \tilde{e}'_A denote the newly created edges running from A to $\text{in}(e)$ and $\text{in}(e')$ (respectively), as depicted in Fig. 9. Note that the orientations of the new edges are inherited from the orientations of the original edge.

We define the map

$$\varrho : (\mathfrak{g} \times \mathfrak{g}) \times G^{E(\Gamma \cup_A \Gamma')} \rightarrow T(G^{E_\Gamma}) \times T(G^{E_{\Gamma'}})$$

by

$$\varrho(\xi, \eta, g)_{e_0} := \begin{cases} (L_{g_{\tilde{e}_A}} R_{g_{e_A}})_* \xi & \text{if } e_0 = e, \\ (L_{g_{\tilde{e}'_A}} R_{g_{e'_A}})_* \eta & \text{if } e_0 = e', \\ g_{e_0} & \text{otherwise.} \end{cases}$$

Where $L_g, R_g : G \rightarrow G$ denote Left, Right multiplication by $g \in G$, and we have identified \mathfrak{g} with the tangent space at the identity. The map ϱ is $G^{V(\Gamma \cup_A \Gamma')}$ -equivariant, where the action is defined on $\mathfrak{g} \times \mathfrak{g}$ by

$$g \cdot (\xi, \eta) := (\mathbf{Ad}_{g_A} \xi, \mathbf{Ad}_{g_A} \eta), \quad g \in G^{V(\Gamma \cup_A \Gamma')}, \quad \xi, \eta \in \mathfrak{g}.$$

The universal property of the tensor product implies that ϱ extends to a map

$$\varrho : (\mathfrak{g} \otimes \mathfrak{g}) \times G^{E(\Gamma \cup_A \Gamma')} \rightarrow T(G^{E_\Gamma}) \otimes T(G^{E_{\Gamma'}}) \subseteq \otimes^2 T(G^{E(\Gamma \cup \Gamma')})$$

We define

$$\Psi_s := \varrho(s, \cdot) : G^{E(\Gamma \cup_A \Gamma')} \rightarrow \otimes^2 T(G^{E(\Gamma \cup \Gamma')}).$$

Since s is G invariant, it follows that Ψ_s is $G^{V(\Gamma \cup_A \Gamma')}$ -equivariant. Therefore

$$(f \otimes f' \rightarrow \Psi_s^*(df \otimes df')) : \mathcal{A}_\Gamma(G) \otimes \mathcal{A}_{\Gamma'}(G) \rightarrow \mathcal{A}_{(\Gamma \cup_A \Gamma')}(G)$$

is a $G_\Gamma^{V_{anch}} \times G_{\Gamma'}^{V_{anch}}$ -equivariant linear map.

Proposition 3. *Let $\text{SpinNet}_{\Sigma, V}(G)$ be endowed with the bracket*

$$(20) \quad \{[\Gamma, f], [\Gamma', f']\} := \sum_{A \in (\Gamma \times_\Sigma \Gamma')^{int}} \text{sign}(A) [\Gamma \cup_A \Gamma', \Psi_s^*(df \otimes df')] \\ + \sum_{A=(v, v') \in (\Gamma \times_\Sigma \Gamma')^{anch}} \frac{1}{2} \text{sign}(A) [\Gamma \cup \Gamma', (df \otimes df')((\rho_v \otimes \rho_{v'})s)],$$

where $\text{sign}(\cdot) = \pm 1$ is computed as pictured in Fig. 4, and Γ and Γ' are assumed to be transverse graph diagrams. Then

$$(21) \quad \pi(\text{dev}(\Gamma, f), \text{dev}(\Gamma', f')) = \text{ev} \{[\Gamma, f], [\Gamma', f']\}.$$

Proof. Suppose first that $V_{\Gamma}^{\text{int}} = \emptyset = V_{\Gamma'}^{\text{int}}$. Then $\mathcal{A}_{\Gamma}(G) = C^{\infty}(G^{E_{\Gamma}})$, and $\text{dev}(\Gamma, f) = \prod_{e \in E_{\Gamma}} \text{hol}_e^* \text{d}f$. Thus

$$(22) \quad \text{dev}(\Gamma, f) = \sum_{e \in E_{\Gamma}} ((L_{\text{hol}_e})_e^* \text{d}f) (\text{hol}_e^* \theta^L)$$

where $(L_g)_e, (R_g)_e : G^{E_{\Gamma}} \rightarrow G^{E_{\Gamma}}$ denotes the left, right multiplication by $g \in G$ on the $e \in E_{\Gamma}$ -th factor of $G^{E_{\Gamma}}$. Substituting Eq. (22) (and the corresponding expression for $\text{dev}(\Gamma', f')$) into Eq. (12) results in the equality

$$\begin{aligned} \pi(\text{dev}(\Gamma, f), \text{dev}(\Gamma', f')) \\ = \frac{1}{2} \sum_{e \in E_{\Gamma}, e' \in E_{\Gamma'}} ((L_{\text{hol}_e})_e^* \text{d}f) \otimes ((L_{\text{hol}_{e'}})_{e'}^* \text{d}f') (\mathbf{A} \text{d}_{\text{hol}_{(e, e')}} \otimes 1) s \end{aligned}$$

Expanding (e, e') in the last line using Eq. (11), and simplifying the resulting expression using the equalities $\text{hol}_e = \text{hol}_{\bar{e}_A} \text{hol}_{e_A}$ and $\text{hol}_{e'} = \text{hol}_{\bar{e}'_A} \text{hol}_{e'_A}$, yields

$$\begin{aligned} \sum_{A \in e \cap e', e \in E_{\Gamma}, e' \in E_{\Gamma'}} \frac{1}{2} \lambda(A) \text{sign}(A) (\text{d}f \otimes \text{d}f') \\ ((L_{\text{hol}_{\bar{e}_A}})_e (R_{\text{hol}_{e_A}})_e \otimes (L_{\text{hol}_{\bar{e}'_A}})_{e'} (R_{\text{hol}_{e'_A}})_{e'}) s). \end{aligned}$$

If $A \in \partial\Sigma$, then $\lambda(A) = 1$, and

$$(L_{\text{hol}_{\bar{e}_A}})_e (R_{\text{hol}_{e_A}})_e = \rho_v, \quad (L_{\text{hol}_{\bar{e}'_A}})_{e'} (R_{\text{hol}_{e'_A}})_{e'} = \rho_{v'},$$

where $A = (v, v') \in (\Gamma \times_{\Sigma} \Gamma')^{\text{anch}}$. On the other hand, if $A \notin \partial\Sigma$, then $\lambda(A) = 2$, and

$$(\text{d}f \otimes \text{d}f') (((L_{\text{hol}_{\bar{e}_A}})_e (R_{\text{hol}_{e_A}})_e \otimes (L_{\text{hol}_{\bar{e}'_A}})_{e'} (R_{\text{hol}_{e'_A}})_{e'}) s) = \Psi_s^*(\text{d}f \otimes \text{d}f').$$

Therefore Eq. (21) holds in this case.

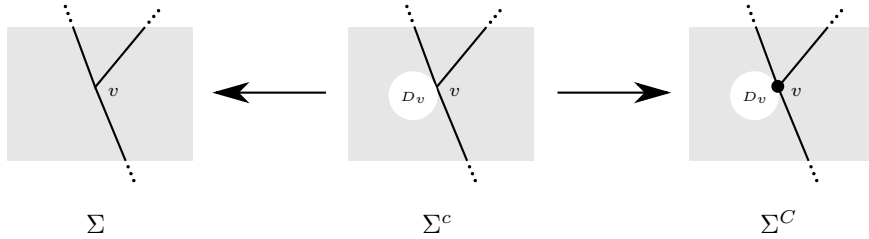


FIGURE 10. For each vertex v , we delete a small open disk, D_v , from Σ such that its boundary ∂D_v intersects the graph precisely at v .

Suppose now that Γ and Γ' are arbitrary graph diagrams (which are transverse to each other). Let Σ^c be the marked surface obtained from Σ as follows: for each $v \in V_{\Gamma}^{\text{int}} \cup V_{\Gamma'}^{\text{int}}$ delete a small open disk $D_v \subset \Sigma \setminus \Gamma \cup \Gamma'$ such that

$$\partial D_v \cap (\Gamma \cup \Gamma') = v.$$

Let Σ^C be the marked surface obtained by Σ^c by adding a marked point at v (cf. Fig. 10).

If we view (Γ, f) and (Γ', f') as spin networks in Σ^C , all their vertices are anchor points. Therefore Eq. (20) holds for these spin networks in Σ^C . Finally, applying Corollary 1 to the embeddings

$$(\Sigma^c, V) \rightarrow (\Sigma, V)$$

and

$$(\Sigma^c, V) \rightarrow (\Sigma^C, V \cup V_{\Gamma}^{int} \cup V_{\Gamma'}^{int})$$

shows that Eq. (20) holds for these spin networks in Σ . \square

8.4. Spin networks on quilted surfaces. Let Σ be a quilted surface.

Definition 7. A graph diagram in Σ consists of the following data:

- a directed graph, Γ with edges E_{Γ} and vertices V_{Γ} ,
- a subset of vertices $V_{\Gamma}^{anch} \subseteq V_{\Gamma}$, called ‘anchor points’, and
- a smooth map $\iota : \Gamma \rightarrow \Sigma$ such that

$$\iota^{-1}(\mathbf{Wall}) = V_{\Gamma}^{anch}.$$

Often we will abuse notation and denote the graph diagram simply by Γ .

A morphism between graph diagrams $\iota : \Gamma \rightarrow \Sigma$ and $\iota' : \Gamma' \rightarrow \Sigma$ is a map $\mu : \Gamma \rightarrow \Gamma'$ such that $\iota = \iota' \circ \mu$ and $\mu(V_{\Gamma}) \subseteq V_{\Gamma'}$.

A homotopy between graph diagrams $\iota_0 : \Gamma \rightarrow \Sigma$ and $\iota_1 : \Gamma \rightarrow \Sigma$ is one parameter family of graph diagrams $\iota_t : \Gamma \rightarrow \Sigma$, $t \in [0, 1]$.

Suppose $\iota : \Gamma \rightarrow \Sigma$ is a graph diagram. We denote by $\tilde{\iota}$ the natural map $V_{\Gamma}^{anch} \rightarrow \mathbf{Wall}$. Let $\Gamma_d = \iota^{-1}(\Sigma_d)$, then $\Gamma_d \rightarrow \Sigma_d$ is a graph diagram in the marked surface (Σ_d, V_d) , (here we have composed with the blow down map $\hat{\Sigma}_d \rightarrow \Sigma_d$). Define

$$\mathcal{A}_{(\iota : \Gamma \rightarrow \Sigma)} := C^{\infty} \left(\prod_{d \in \mathbf{Dom}} G_d^{E_{\Gamma_d}} \right)^{C_{\Gamma}^{int} \times C_{\Gamma}^{anch}},$$

where

$$C_{\Gamma}^{int} := \prod_{d \in \mathbf{Dom}} G_d^{V_{\Gamma_d}^{int}}, \quad C_{\Gamma}^{anch} := \prod_{v \in V_{\Gamma}^{anch}} C_{\tilde{\iota}(v)},$$

with $V_{\Gamma_d}^{int} := V_{\Gamma_d} \setminus V_{\Gamma_d}^{anch}$. Notice that

$$(23) \quad \mathcal{A}_{(\iota : \Gamma \rightarrow \Sigma)} = \left(\hat{\otimes}_{d \in \mathbf{Dom}} \mathcal{A}_{\Gamma_d}(G_d) \right)^{C_{\Gamma}^{anch}}.$$

To simplify notation, we will often abbreviate $\mathcal{A}_{(\iota : \Gamma \rightarrow \Sigma)}$ to \mathcal{A}_{Γ} .

As in the case of spin networks in marked surfaces, we the following lemma.

Lemma 5. The following two facts hold:

- The algebras $\mathcal{A}_{(\iota_0 : \Gamma \rightarrow \Sigma)}$ and $\mathcal{A}_{(\iota_1 : \Gamma \rightarrow \Sigma)}$ are canonically identified for homotopic graph diagrams $\iota_0 : \Gamma \rightarrow \Sigma$ and $\iota_1 : \Gamma \rightarrow \Sigma$.
- The assignment $(\iota : \Gamma \rightarrow \Sigma) \mapsto \mathcal{A}_{(\iota : \Gamma \rightarrow \Sigma)}$ is a functor from graph diagrams to algebras.

Definition 8. A spin network in Σ is a pair (Γ, f) , where $\Gamma \rightarrow \Sigma$ is a graph diagram, and $f \in \mathcal{A}_{\Gamma}$.

We say that spin networks (Γ_0, f_0) and (Γ_1, f_1) are homotopic if the underlying graph diagrams are homotopic and $f_0 = f_1 \in \mathcal{A}_{\Gamma_0} \equiv \mathcal{A}_{\Gamma_1}$.

A morphism of spin networks $\mu : (\Gamma, f) \rightarrow (\Gamma', f')$ is a morphism of graph diagrams $\mu : \Gamma \rightarrow \Gamma'$ such that $f' = \mu_* f$.

Let

$$\mathcal{M}_\Sigma := \left(\prod_{d \in \mathbf{Dom}} M_{\Sigma_d, V_d}(G_d) \right) / C$$

denote the moduli space for the quilted surface, Σ . Then

$$C^\infty(\mathcal{M}_\Sigma) = C^\infty \left(\prod_{d \in \mathbf{Dom}} M_{\Sigma_d, V_d}(G_d) \right)^C = \left(\hat{\otimes}_{d \in \mathbf{Dom}} C^\infty(M_{\Sigma_d, V_d}(G_d)) \right)^C$$

Since the map

$$\text{ev} : \hat{\otimes}_{d \in \mathbf{Dom}} \mathcal{A}_{\Gamma_d}(G_d) \rightarrow \hat{\otimes}_{d \in \mathbf{Dom}} C^\infty(M_{\Sigma_d, V_d}(G_d))$$

is $\prod_{d \in \mathbf{Dom}} G_d^{V_d}$ -equivariant it restricts to a map of algebras

$$\text{ev} : \mathcal{A}_\Gamma \rightarrow C^\infty(\mathcal{M}_\Sigma).$$

As before, we consider two spin networks to be equivalent if they are related by a chain of homotopies and morphisms, and we define SpinNet_Σ to be the set of equivalence classes of spin networks in the quilted surface, Σ . The same arguments as in Section 8.2 shows that ev descends to a map of equivalence classes:

$$\text{ev} : \text{SpinNet}_\Sigma \rightarrow C^\infty(\mathcal{M}_\Sigma).$$

Moreover, the analogues of Lemma 4, Proposition 2, and Proposition 3 hold for SpinNet_Σ .

To be precise, suppose that $\iota : \Gamma \rightarrow \Sigma$ and $\iota' : \Gamma' \rightarrow \Sigma$ are two graph diagrams which are transverse. That is, $\iota(V_\Gamma) \cap \iota'(V_{\Gamma'}) = \emptyset$ and the restrictions of ι and ι' to the edges are transverse. Suppose $A \in \Gamma \times_\Sigma \Gamma'$ and let $d_A \in \mathbf{Dom}$ denote the domain such that $A \in \Sigma_{d_A}$. We define the graph diagram $\iota \cup_A \iota' : \Gamma \cup_A \Gamma' \rightarrow \Sigma$ as in Section 8.3, and we define

$$\Psi : \prod_{d \in \mathbf{Dom}} G_d^{E_{(\Gamma \cup \Gamma')_d}} \rightarrow \otimes^2 T \left(\prod_{d \in \mathbf{Dom}} G_d^{E_{(\Gamma \cup \Gamma')_d}} \right)$$

to be the sum of $\Psi_{s_{d_A}}$ on the d_A -th factor with the zero sections on the other factors.

Theorem 7. *The map*

$$\text{ev} : \text{SpinNet}_\Sigma \rightarrow C^\infty(\mathcal{M}_\Sigma)$$

is an isomorphism of Poisson algebras, where scalar multiplication, addition, and multiplication are defined on SpinNet_Σ by Eqs. 18, and the Poisson bracket is defined by

$$(24) \quad \{[\Gamma, f], [\Gamma', f']\} := \sum_{A \in \Gamma \times_\Sigma \Gamma'} \text{sign}(A) [\Gamma \cup_A \Gamma', \Psi^*(df \otimes df')],$$

where Γ and Γ' are assumed to be transverse graph diagrams.

Proof. The proof that $\text{ev} : \text{SpinNet}_\Sigma \rightarrow C^\infty(\mathcal{M}_\Sigma)$ is an isomorphism of algebras is entirely analogous to that of Proposition 2, and so we omit it.

As explained in Section 8.3, Ψ is equivariant with respect to the action of $\prod_{d \in \mathbf{Dom}} G_d^{V_{(\Gamma \cup \Gamma')_d}}$. In particular, $\Psi^*(df \otimes df') \in \mathcal{A}_{(\Gamma \cup \Gamma')}$.

Equation (23) identifies SpinNet_Σ with the subalgebra of C -invariant elements of

$$\hat{\otimes}_{d \in \mathbf{Dom}} \text{SpinNet}_{\Sigma_d, V_d}(G_d).$$

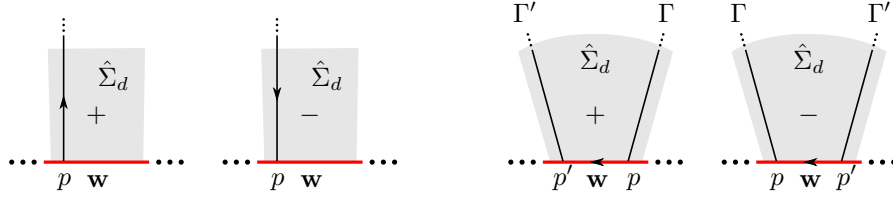
Moreover, the following diagram commutes:

$$\begin{array}{ccc} \text{SpinNet}_\Sigma & \xrightarrow{\text{ev}} & C^\infty(\mathcal{M}_\Sigma) \\ \downarrow & & \downarrow \\ \bigotimes_{d \in \mathbf{Dom}} \text{SpinNet}_{\Sigma_d, V_d}(G_d) & \xrightarrow{\text{ev}} & C^\infty\left(\prod_{d \in \mathbf{Dom}} M_{\Sigma_d, V_d}(G_d)\right) \end{array}$$

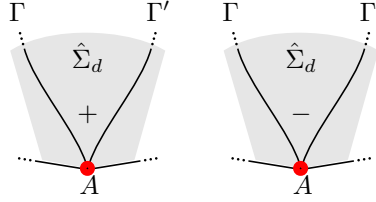
Hence Theorem 1.A and Proposition 3 imply that Eq. (20) defines a Poisson bracket on SpinNet_Σ for which ev is a morphism of Poisson algebras. Explicitly, this bracket is

$$\begin{aligned} \{[\Gamma, f], [\Gamma', f']\} &:= \sum_{d \in \mathbf{Dom}} \left(\sum_{A \in (\Gamma \times_\Sigma \Gamma')_d^{\text{int}}} \text{sign}(A) [\Gamma \cup_A \Gamma', \Psi^*(df \otimes df')] \right. \\ &\quad \left. + \sum_{A=(p, p') \in (\Gamma_d \times_{\Sigma_d} \Gamma'_d)^{\text{anch}}} \frac{1}{2} \text{sign}(A) [\Gamma \cup \Gamma', (df \otimes df')((\rho_p \otimes \rho_{p'})s_d)] \right). \end{aligned}$$

The first term simplifies to yield Eq. (24), so we need only show that the second term vanishes.



(A) $\text{Sign}(p, d) = \pm 1$ depends on the direction of the edge in Γ leaving p . (B) $\text{Sign}(p, p', \mathbf{w}, d) = \pm 1$ depends on the order of p and p' , where we give $\mathbf{w} \subseteq \partial \hat{\Sigma}_d$ the boundary orientation.



(C) The blow down map $\hat{\Sigma}_d \rightarrow \Sigma_d$ contracts \mathbf{w} to a point. Thus the blow down equates $\epsilon(p, p', \mathbf{w}, d)$ with $\text{sign}(A)$, where the latter is computed as in Fig. 4

FIGURE 11

In terms of graph diagrams in the quilted surface, Σ , the second term can be rewritten as

$$\frac{1}{2} \sum_{\mathbf{w} \in \mathbf{Wall}} \sum_{p \in \mathbf{w} \cap \Gamma, p' \in \mathbf{w} \cap \Gamma'} \sum_{d \in \mathbf{Dom}(\mathbf{w})} \epsilon(p, p', \mathbf{w}, d) [\Gamma \cup \Gamma', (df \otimes df')((\rho_p \otimes \rho_{p'})s_d)]$$

where

$$\mathbf{Dom}(\mathbf{w}) = \{d \in \mathbf{Dom}; \mathbf{w} \subset \partial \hat{\Sigma}_d\}.$$

Both

$$\epsilon(p, p', \mathbf{w}, d) = \text{sign}(p, d) \text{sign}(p', d) \text{sign}(p, p', \mathbf{w}, d),$$

and $\text{sign}(p, d) = \pm 1$ and $\text{sign}(p, p', \mathbf{w}, d) = \pm 1$ are computed as in Fig. 11. Note that $\epsilon(p, p', \mathbf{w}, d) = \text{sign}(A)$, where the left hand side is computed in the blow up, $\hat{\Sigma}_d$, and the right hand side is computed for $A = (p, p')$ in the blow down, Σ_d .

Now both f and f' are $C_{\mathbf{w}}$ -invariant, that is

$$\bigoplus_{d \in \text{Dom}(\mathbf{w})} \rho_p^* df \in \mathfrak{c}_{\mathbf{w}}^\perp, \quad \bigoplus_{d \in \text{Dom}(\mathbf{w})} \rho_{p'}^* df' \in \mathfrak{c}_{\mathbf{w}}^\perp$$

Since $C_{\mathbf{w}}$ was chosen to be coisotropic, $\sum_{d \in \text{Dom}(\mathbf{w})} \epsilon(p, p', \mathbf{w}, d) s_d$ vanishes on $\mathfrak{c}_{\mathbf{w}}^\perp$. This completes the proof. \square

Suppose Σ and Σ' are two quilted surfaces. A map $\Sigma \rightarrow \Sigma'$ is called an *embedding of quilted surfaces* if

- it is an embedding of surfaces $\Sigma \rightarrow \Sigma'$,
- it maps each domain, Σ_d , of Σ into a domain, $\Sigma'_{d'}$, of Σ' which is colored with the same structure group (i.e. $G_d \equiv G_{d'}$), and
- it maps each domain wall, \mathbf{w} , of Σ into a domain wall, \mathbf{w}' , of Σ' which is colored with the same coisotropic relation (i.e. $C_{\mathbf{w}} \equiv C_{\mathbf{w}'}$).

Corollary 3. *An embedding of quilted surfaces $\Sigma \rightarrow \Sigma'$ induces a Poisson morphism*

$$\mathcal{M}_{\Sigma'} \rightarrow \mathcal{M}_{\Sigma}$$

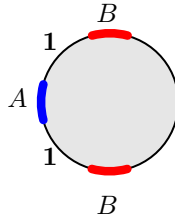
of the corresponding moduli spaces.

Proof. This follows directly from the functoriality of the assignment $\Sigma \rightarrow \text{SpinNet}_{\Sigma}$ of the Poisson algebra of spin networks to quilted surfaces and Theorem 7.

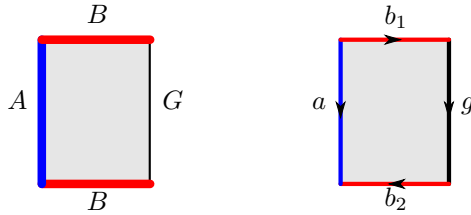
Alternatively, it follows from Corollary 1 and the definition of the Poisson structure on \mathcal{M}_{Σ} given in Eq. (15). \square

9. COLORFUL EXAMPLES

Example 7 (Poisson Lie groups). Suppose that $(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ is a Manin triple and consider the quilted surface Σ pictured below



where $A, B \subset G$ are Lie groups integrating \mathfrak{a} and \mathfrak{b} such that $A \cap B = \mathbf{1}$. Constraining the holonomies along the uncolored boundary arcs marked by $\mathbf{1}$ to be trivial is effectively the same as contracting those arcs to points, which results in the first image below.



Following Theorem 6.B, the moduli space is

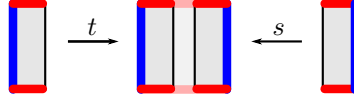
$$\{(b_1, g, b_2, a) \in B \times G \times B \times A; b_2 g b_1 = a\} / (\hat{H} \cap \hat{C}),$$

where the holonomies are as pictured above. Since $A \cap B = \mathbf{1}$, the group $(\hat{H} \cap \hat{C}) \cong B \times B$ of residual gauge transformations acts non-trivially only at the right two vertices by

$$(b, b') \cdot (b_1, g, b_2, a) = (bb_1, b'gb^{-1}, b_2b'^{-1}, a).$$

Thus the moduli space is identified with A . The Poisson structure on A is the Poisson Lie structure. This example should be compared with Example 4.

Consider the embedding of quilted surfaces pictured below:



As explained above, the moduli space corresponding to the quilted surfaces depicted on the left and right is the Poisson Lie group, A . Meanwhile, as explained in Example 6, the moduli space for the quilted surface depicted in the middle is the symplectic groupoid

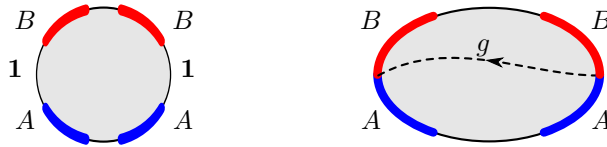
$$\{(a_1, b_1, a_2, b_2) \in A \times B \times A \times B; a_1 b_1 = b_2 a_2\},$$

integrating the Poisson Lie structure on A . The embedding of quilted surfaces induces the map

$$t \times s : (a_1, b_1, a_2, b_2) \rightarrow (a_2, a_1).$$

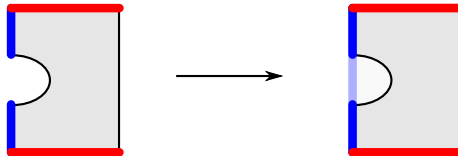
The first component of the map is the Lie groupoid target map, while the second component is the Lie groupoid source map. Corollary 3 shows that this map is a Poisson morphism.

Example 8 (Double Poisson Lie group). Suppose once again that $(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ is a Manin triple, and that $A, B \subset G$ are Lie groups integrating \mathfrak{a} and \mathfrak{b} such that $A \cap B = \mathbf{1}$. Then, G is a Poisson Lie group (cf. [10], see also [17], for example), we may describe this Poisson structure in terms of a moduli space as follows: Let Σ be the quilted surface pictured below left,



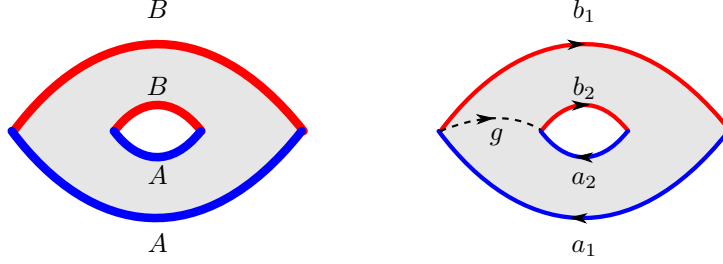
Contracting the uncolored boundary arcs marked by $\mathbf{1}$ results in the image on the right. Since $A \cap B = \mathbf{1}$, the group $(\hat{H} \cap \hat{C})$ of residual gauge transformations is trivial (cf. Theorem 6.B). Consequently, computing the holonomy along the dotted arc pictured in the second image, we may identify the moduli space with G . The resulting Poisson structure on G is Drinfel'd's Double Poisson Lie structure.

Applying Corollary 3 to the following embedding of quilted surfaces



yields the morphism of Poisson Lie groups $A \rightarrow G$.

Example 9 (Symplectic double groupoid integrating Drinfel'd's double [23, 25]). Suppose once again that $(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ is a Manin triple, and that $A, B \subset G$ are Lie groups integrating \mathfrak{a} and \mathfrak{b} such that the product map $A \times B \rightarrow G$ is a diffeomorphism. We may identify the symplectic double groupoid integrating the Poisson Lie structure on G (described in the previous example) with a moduli space, as follows: Let Σ be the quilted surface pictured below, where we have already contracted all the uncolored boundary arcs.

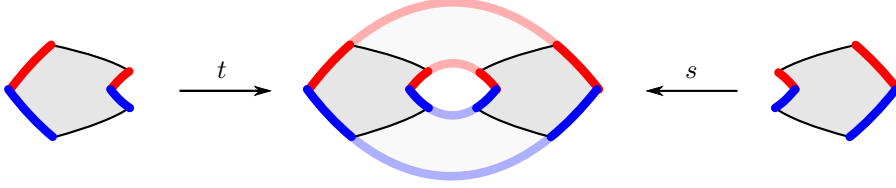


The holonomies along the arcs pictured above identify the moduli space with

$$\{(a_1, b_1, a_2, b_2, g) \in A \times B \times A \times B \times G; ga_1b_1 = a_2b_2g\},$$

where once again, the group of residual gauge transformations is trivial.

Applying Corollary 3 to the following embedding of quilted surfaces



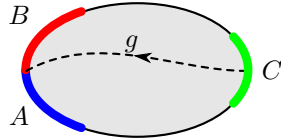
yields the Poisson morphism

$$t \times s : (a_1, b_1, a_2, b_2, g) \rightarrow (g, b_2gb_1^{-1})$$

whose components are the target/source map (respectively) onto G endowed with the Poisson Lie structure described in Example 8.

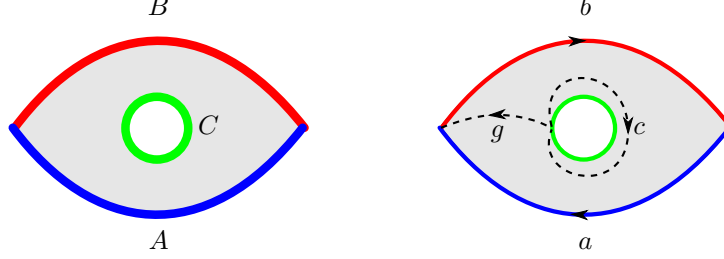
This moduli space was first studied in [23, 25].

Example 10 (Lu-Yakimov Poisson Homogeneous spaces). Suppose once again that $(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ is a Manin triple, and that $A, B \subset G$ are Lie groups integrating \mathfrak{a} and \mathfrak{b} such that $A \cap B = \mathbf{1}$. Let $C \subseteq G$ be a closed subgroup whose Lie subalgebra $\mathfrak{c} \subseteq \mathfrak{g}$ is coisotropic. Lu and Yakimov [18] describe a Poisson structure on G/C , which we may identify with the moduli space for the following quilted surface:



Computing the holonomy along the dotted arc yields an element $g \in G$, but the group C of residual gauge transformations acts by right multiplication on this element. Thus, following Theorem 6.B, the moduli space G/C carries a Poisson structure.

The symplectic groupoid integrating the Poisson structure on G/C is the moduli space corresponding to the quilted surface pictured below:



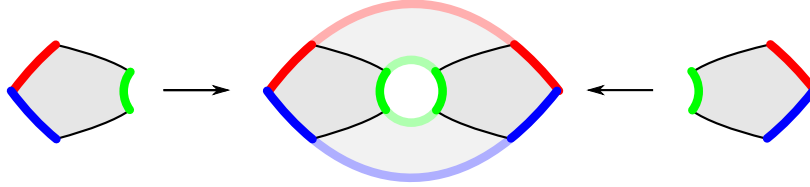
The holonomies along the arcs pictured above identify the moduli space with

$$\{(a, b, c, g) \in A \times B \times C^\perp \times G; \text{ } abg = gc\}/C$$

where $c' \in C$ acts by

$$c' \cdot (a, b, c, g) = (a, b, c'cc'^{-1}, gc'^{-1}).$$

Applying Corollary 3 to the embedding of quilted surfaces pictured below



yields the following source and target maps onto G/C :

$$t \times s : (a, b, c, g) \rightarrow (g, bg).$$

Example 11 (Fission spaces [7]). Suppose that $s \in S^2(\mathfrak{g})^\mathfrak{g}$ is non-degenerate, $\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{h} \oplus \mathfrak{u}_-$ as a vector space (but not as a Lie algebra), where $\mathfrak{p}_\pm := \mathfrak{h} \oplus \mathfrak{u}_\pm \subseteq \mathfrak{g}$ are coisotropic subalgebras satisfying $\mathfrak{p}_\pm^\perp = \mathfrak{u}_\pm$. Suppose further that the Lie subalgebras $\mathfrak{u}_\pm, \mathfrak{p}_\pm, \mathfrak{h}$ all integrate to closed subgroups $U_\pm, P_\pm, H \subseteq G$ such that $H = P_+ \cap P_-$. The metric on \mathfrak{g} descends to a non-degenerate invariant metric on $\mathfrak{h} \subseteq \mathfrak{g}$, and

$$(25) \quad C_\pm := \{(c, \tilde{c}) \in H \times G; \tilde{c}c^{-1} \in U_\pm\}$$

is a coisotropic subgroup of $H \times G$.

Consider the quilted surface pictured in Fig. 12. Computing holonomies along the dashed lines yields

$$\{(h, h_0, \dots, h_{2r-1}; C_0, C_1, \dots, C_{2r}) \in H^{2r+1} \times G^{2r+1} \\ ; h_{2i+1}^{-1} C_{2i+1} C_{2i}^{-1} h_{2i} \in U_+ \text{ and } h_{2i}^{-1} C_{2i} C_{2i-1}^{-1} h_{2i-1} \in U_-, \}.$$

Meanwhile, since $P_+ \cap P_- = G$, the group of residual gauge transformations is $\prod_{x_1, \dots, x_{2r}} H$, acting at the appropriate points on the quilted surface. Thus, up to a gauge transformation, we may assume that $h_0 = h_1 = \dots = h_{2r-1} = \mathbf{1}$. Setting $S_i = C_i C_{i-1}^{-1}$, we see that that the moduli space can be identified with

$${}_G \mathcal{A}_H^r := \{(h; S_{2r}, \dots, S_1; C_0) \in H \times (U_- \times U_+)^r \times G\}.$$

Theorem 6.C implies that ${}_G \mathcal{A}_H^r$ is a quasi-Poisson $G \times H$ manifold, where $g \in G$ and $k \in H$ act at the marked points v_G and v_H , respectively:

$$(g, k) \cdot (h; S_{2r}, \dots, S_1; C_0) = (khk^{-1}, kS_{2r}k^{-1}, \dots, kS_1k^{-1}, kC_0g^{-1}),$$

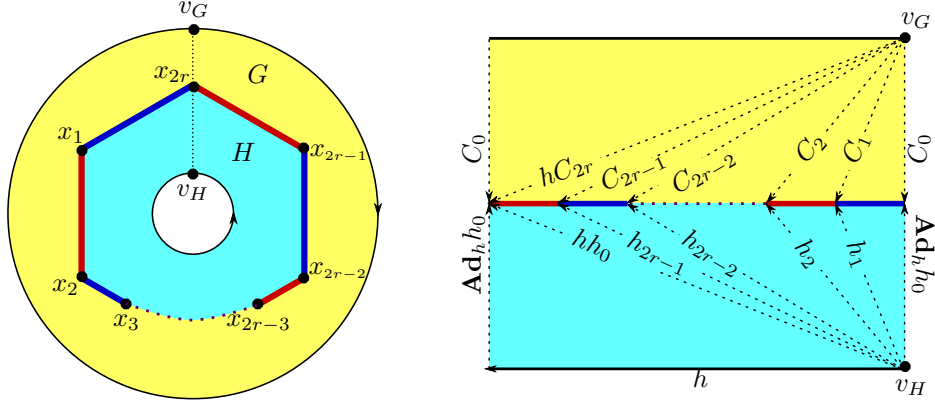


FIGURE 12. On the surface pictured above, the structure group in the yellow domain is G while the structure group in the blue domain is H . Along the boundary of the two domains, blue edges are colored with C_+ while the red edges are colored with C_- . The dotted segment of the boundary between the two annuli is meant to indicate an alternating sequence of blue and red edges. Cutting along the vertical dotted line in the first picture yields the second picture. Acting by H at the points x_1, \dots, x_{2r} allows us to set the holonomies h_0, \dots, h_{2r-1} to the identity.

The holonomy along the boundary components,

$$(h; S_{2r}, \dots, S_1; C_0) \rightarrow (C_0^{-1} h S_{2r} \cdots S_1 C_0, h^{-1}),$$

defines a moment map on ${}_G \mathcal{A}_H^r$.

This quasi-Hamiltonian $G \times H$ -space was first discovered by Boalch [7, 8, 9], who used it to study meromorphic connections on Riemann surfaces.

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